

A NORMAL FORM THEOREM AROUND SYMPLECTIC LEAVES

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ABSTRACT. We prove the Poisson geometric version of the Local Reeb Stability (from foliation theory) and of the Slice Theorem (from equivariant geometry), which is also a generalization of Conn's linearization theorem.

INTRODUCTION

Recall that a **Poisson structure** on a manifold M is a Lie bracket $\{\cdot, \cdot\}$ on the space $C^\infty(M)$ of smooth functions on M which acts as a derivation in each entry

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \quad (\forall) f, g, h \in C^\infty(M).$$

A Poisson structure can be given also by a bivector $\pi \in \mathfrak{X}^2(M)$, involutive with respect to the Schouten bracket, i.e. $[\pi, \pi] = 0$; one has:

$$\langle \pi, df \wedge dg \rangle = \{f, g\}, \quad (\forall) f, g \in C^\infty(M).$$

To each function $f \in C^\infty(M)$ one assigns the Hamiltonian vector field

$$X_f = \{f, \cdot\} \in \mathfrak{X}(M).$$

The flows of the Hamiltonian vector fields give a partition of M into **symplectic leaves**; they carry a canonical smooth structure, which makes them into regular immersed submanifolds, whose tangent spaces are spanned by the Hamiltonian vector fields; each leaf S is a symplectic manifold, with the symplectic structure:

$$\omega_S(X_f, X_g) = \{f, g\}.$$

In this paper we prove a normal form theorem around symplectic leaves, which generalizes Conn's linearization theorem [5] (for 1-point leaves) and is a Poisson geometric analogue of the local Reeb stability from foliation theory and of the slice theorem from group actions. We will use the Poisson homotopy bundle of a leaf S , which is the analogue of to the holonomy cover from foliation theory

$$P_x \longrightarrow S$$

($x \in S$ a base point), whose "structural group" is the Poisson homotopy group G_x .

Main Theorem 1. *Let (M, π) be a Poisson manifold and let S be a compact leaf. If the Poisson homotopy bundle over S is a smooth compact manifold with vanishing second DeRham cohomology group, then, in a neighborhood of S , π is Poisson diffeomorphic to its first order model around S .*

A detailed statement and reformulations appear in section 2. The proof uses ideas similar to the ones in [11]: a Moser-type argument reduces the problem to a cohomological one (Theorem 4.1); a Van Est argument and averaging reduces the cohomological problem to an integrability problem (Theorem 5.1) which, in turn, can be reduced to the existence of special symplectic realization (Theorem 5.2); the symplectic realization is then built by working on the Banach manifold of cotangent paths (subsection 5.4). For an outline of the paper, we advise the reader to go through the introductory sentence(s) of each section.

There have been various attempts to generalize Conn's linearization theorem to arbitrary symplectic leaves. While the desired conclusion was clear (the same as in our theorem), the assumptions (except for the compactness of S) are more subtle. Of course, as for any (first order) local form result, one looks for assumptions on the first jet of π along S . Here are a few remarks on the assumptions.

1. *Compactness assumptions.* It was originally believed that such a result could follow by first applying Conn's theorem to a transversal to S . Hence the expected assumption was, next to the compactness of S , that the isotropy Lie algebra \mathfrak{g}_x ($x \in S$) is semi-simple of compact type. The failure of such a result was already pointed out in [12]. A refined conjecture was made in [9]- revealing the compactness assumptions that appear in our theorem. The idea is the following: while the condition that \mathfrak{g}_x is semi-simple of compact type is equivalent to the fact that all (connected) Lie groups integrating the Lie algebra \mathfrak{g}_x are compact, one should require the compactness (and smoothness) of only one group associated to \mathfrak{g}_x - the Poisson homotopy group G_x . This is an important difference because

- our theorem may be applied even when \mathfrak{g}_x is abelian.
- actually, under the assumptions of the theorem, \mathfrak{g}_x can be semi-simple of compact type only when the leaf is a point!

2. *Vanishing of $H^2(P_x)$.* The compactness condition on the Poisson homotopy bundle is natural also when drawing an analogy with other local normal form results like local Reeb stability or the slice theorem. However, compactness alone is not enough (see Example 2.4). The subtle condition is $H^2(P_x) = 0$ and its appearance is completely new in the context of normal forms:

- In Conn's theorem, it is not visible (it is automatically satisfied!).
- In the classical cases (foliations, actions) such a condition is not needed.

What happens is that the vanishing condition is related to integrability phenomena [7, 8]. In contrast with the case of foliations and of group actions, Poisson manifolds give rise to Lie algebroids that may fail to be integrable. To clarify the role of this assumption, we mention here:

It implies integrability. The main role of this assumption is that it forces the Poisson manifold to be (Hausdorff) integrable around the leaf. Actually, under such an integrability assumption, the normal form is much easier to establish, and the vanishing condition is not needed- see our Proposition 2.1, which can also be deduced from Zung's linearization theorem [32]. Note however that such an integrability condition refers to the germ of π around S (and not the first order jet, as desired!); and, of course, Conn's theorem does not make such an assumption.

It implies vanishing of the second Poisson cohomology. Next to integrability, the vanishing condition also implies the vanishing of the second Poisson cohomology group $H_\pi^2(U)$ (of arbitrarily small neighborhoods U of S)- which is known to be relevant to infinitesimal deformations (see e.g. [9]). We would like to point out that the use of $H_\pi^2(U) = 0$ only simplifies our argument but is not essential. A careful analysis shows that one only needs a certain class in $H_\pi^2(U)$ to vanish, and this can be shown using only integrability. This is explained at the end of subsection 5.1, when concluding the proof of Proposition 2.1 mentioned above.

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1. A MORE DETAILED INTRODUCTION

In this section we give more details on the statement of the main theorem. We start by recalling some classical normal form theorems in differential geometry. Then we discuss the local model associated to a principal bundle over a symplectic manifold. Next we describe the Poisson homotopy bundle in detail and we finish the section with an overview of the notion of integrability of Poisson manifolds.

1.1. The Slice Theorem. Let G be Lie group acting on a manifold M , $x \in M$ and denote by \mathcal{O} the orbit through x . The Slice Theorem ([14]) gives a normal form for the G -manifold M around \mathcal{O} . It is built out of the isotropy group G_x at x and its canonical representation $V_x = T_x M / T_x \mathcal{O}$. Explicitly, the local model is:

$$G \times_{G_x} V_x = (G \times V_x) / G_x$$

which is a G -manifold and admits \mathcal{O} as the orbit corresponding to $0 \in V_x$.

Theorem 1.1. *If G is compact, then a G -invariant neighborhood of \mathcal{O} in M is diffeomorphic, as a G -manifold, to a G -invariant neighborhood of \mathcal{O} in $G \times_{G_x} V_x$.*

It is instructive to think of the building pieces of the local model as a triple $(G_x, G \rightarrow \mathcal{O}, V_x)$ consisting of the Lie group G_x , the principal G_x -bundle G over \mathcal{O} and a representation V_x of G_x . This triple should be thought of as the first order data (first jet) along \mathcal{O} associated to the G -manifold M , while of the associated local model as the first order approximation.

1.2. Local Reeb stability. Let \mathcal{F} be a foliation on a manifold M , $x \in M$ and denote by L the leaf through x . The Local Reeb Stability Theorem ([25]) gives a normal form for the foliation around L (we state below a weaker version). Denote by \tilde{L} the universal cover of L , and consider the linear holonomy representation of $\Gamma_x := \pi_1(L, x)$ on $N_x = T_x M / T_x L$. The local model is $\tilde{L} \times_{\Gamma_x} N_x$ with leaves $\tilde{L} \times_{\Gamma_x} (\Gamma_x v)$ for $v \in N_x$; L corresponds to $v = 0$.

Theorem 1.2. *If L is compact and Γ_x is finite, then a saturated neighborhood of L in M is diffeomorphic, as a foliated manifold, to a neighborhood of L in $\tilde{L} \times_{\Gamma_x} N_x$.*

Again, the local model is build out of a triple $(\Gamma_x, \tilde{L} \rightarrow L, N_x)$, consisting of the discrete group Γ_x , the principal Γ_x -bundle \tilde{L} and a representation N_x of Γ_x . The triple should be thought of as the first order data along L associated to the foliated manifold M and the local model as the first order approximation.

1.3. Conn's Linearization Theorem. Let (M, π) be a Poisson manifold and $x \in M$ be a zero of π . Conn's theorem [5] gives a normal form for (M, π) near x , built out of the isotropy Lie algebra \mathfrak{g}_x . Recall that $\mathfrak{g}_x = T_x^* M$ with the bracket:

$$(1) \quad [d_x f, d_x g] = d_x \{f, g\}, \quad f, g \in C^\infty(M).$$

Conversely, there is a Poisson bracket π_{lin} on the dual \mathfrak{g}^* of any Lie algebra:

$$\pi_{\text{lin}}(X, Y)_\xi := \langle \xi, [X, Y] \rangle, \quad (\forall) \xi \in \mathfrak{g}^*, \quad X, Y \in \mathfrak{g} = T_\xi^* \mathfrak{g}^*.$$

Theorem 1.3. *If \mathfrak{g}_x is semi-simple of compact type then a neighborhood of x in M is Poisson-diffeomorphic to a neighborhood of the origin in $(\mathfrak{g}_x^*, \pi_{\text{lin}})$.*

Again, the local data (the Lie algebra \mathfrak{g}_x) should be viewed as the first order data at x associated to the Poisson manifold, while the local model as the first order approximation. To make the analogy with the previous two theorems, we replace the Lie algebra \mathfrak{g}_x by $G_x := G(\mathfrak{g}_x)$, the 1-connected Lie group integrating \mathfrak{g}_x . The local data is then

$$(G_x, G_x \rightarrow \{x\}, \mathfrak{g}_x^*)$$

and the local model is defined on $G_x \times_{G_x} \mathfrak{g}_x^* = \mathfrak{g}_x^*$. The most convincing argument for bringing G_x into the picture is the fact that the assumption of the theorem is equivalent to the fact that G_x is compact.

1.4. The local model. In this subsection we explain the local model. The construction given below is standard in symplectic geometry and goes back to the local forms of Hamiltonian spaces around the level sets of the moment map (cf. e.g. [18]) and also shows up in the work of Montgomery [26] (see also [17]).

The starting data is again a triple consisting of a symplectic manifold (S, ω_S) , which will be our symplectic leaf, a principal G -bundle P over S , which will be the Poisson homotopy bundle and the coadjoint action of G on \mathfrak{g}^* :

$$(G, P \longrightarrow (S, \omega_S), \mathfrak{g}^*).$$

As before, G acts diagonally on $P \times \mathfrak{g}^*$. As a manifold, the local model is:

$$P \times_G \mathfrak{g}^* = (P \times \mathfrak{g}^*)/G.$$

To describe the Poisson structure, we choose a connection 1-form on P , $\theta \in \Omega^1(P, \mathfrak{g})$. The G -equivariance of θ implies that the 1-form $\tilde{\theta}$ on $P \times \mathfrak{g}^*$ defined by

$$\tilde{\theta}_{(p, \mu)} = \langle \mu, \theta_p \rangle$$

is G -invariant. Consider now the G -invariant 2-form

$$\Omega := p^*(\omega_S) - d\tilde{\theta} \in \Omega^2(P \times \mathfrak{g}^*).$$

The open set $M \subset P \times \mathfrak{g}^*$ where it is non-degenerate contains $P \times \{0\}$, therefore (M, Ω) is a symplectic manifold on which G acts freely, in a Hamiltonian fashion, with moment map given by the second projection. Hence $N = M/G \subset P \times_G \mathfrak{g}^*$ inherits a Poisson structure π_N . Notice that S sits as a symplectic leaf in (N, π_N) :

$$(S, \omega_S) = (P \times \{0\}, \Omega|_{P \times \{0\}})/G.$$

Definition 1.4. A *Poisson neighborhood of S in $P \times_G \mathfrak{g}^*$* is any Poisson structure of the type just described, defined on a neighborhood N of S .

Note that different connections induce Poisson structures which have Poisson-diffeomorphic open neighborhoods of S . Also, intuitively, π_N is constructed by combing the canonical Poisson structure on \mathfrak{g}^* with the pullback of the symplectic structure ω_S to P . For instance, if P is trivial, using the canonical connection the resulting model is $(S, \omega) \times (\mathfrak{g}^*, \pi_{\text{lin}})$. Note also that, when P is compact, one can find small enough Poisson neighborhoods of type:

$$N = P \times_G V$$

with $V \subset \mathfrak{g}^*$ a G -invariant open containing 0. Moreover, the resulting symplectic leaves do not depend, as manifolds, on θ . Denoting by $\mathcal{O}_\xi := G\xi$, these are:

$$P/G_\xi \cong P \times_G \mathcal{O}_\xi \subset P \times_G V, \quad \xi \in V.$$

Example 1.5. To understand the role of the bundle P it is instructive to look at the case when $G = T^q$ is a q -torus. As a foliated manifold, the local model is:

$$P \times_G \mathfrak{g}^* = S \times \mathbb{R}^q = \bigcup_t S \times \{t\} \quad (t = (t_1, \dots, t_q) \in \mathbb{R}^q).$$

To complete the description of the local model as a Poisson manifold, we need to specify the symplectic forms ω_t on S - and this is where P comes in. Principal T^q -bundles are classified by q integral cohomology classes $c_1, \dots, c_q \in H^2(S)$; the choice of the connection θ above corresponds to the choice of representatives $\omega_1, \dots, \omega_q \in \Omega^2(S)$ and the resulting Poisson structure corresponds to

$$\omega_t = \omega_S + t_1\omega_1 + \dots + t_q\omega_q.$$

Remark 1. Dirac geometry (see [3] for the basic definitions) provides further insight into our construction. Recall that one of the main features of Dirac structures is that, although they generalize closed 2-forms, they can be pushed-forward. In particular, our 2-form Ω can be pushed-forward to give a Dirac structure $L(\theta)$ on the entire space $P \times_G \mathfrak{g}^*$. Another feature of Dirac structure is that they generalize Poisson bivectors; actually, for a general Dirac structure $L \subset TM \oplus T^*M$ on M , one can talk about the largest open on which L is Poisson (Poisson support of L):

$$\text{sup}(L) := \{x \in M : pr_2(L_x) = T_x^*M\}.$$

Our local model arises from the fact that S is inside the support of $L(\theta)$. Also the independence of θ fits well in this context: if θ' is another connection, then $L(\theta')$ is the gauge transform of $L(\theta)$ with respect to $d(\tilde{\theta} - \tilde{\theta}')$. A simple version of Moser's Lemma can be used to show that $L(\theta)$ and $L(\theta')$ are isomorphic around S .

1.5. The Poisson homotopy bundle I: via cotangent paths. For the statement of the main theorem, we still have to discuss the Poisson homotopy bundle over a symplectic leaf. In this subsection we provide a first description, completely analogous to the construction of the universal cover of a manifold. It is based on the idea that Poisson geometry is governed by “contravariant geometry” - for which we use [8, 15] as references. We recall here a few basic facts. The overall idea is that, in Poisson geometry, the relevant directions are the “cotangent ones”, i.e., for a Poisson manifold (M, π) , one should replace the tangent bundle TM by the cotangent one T^*M . The two are related by

$$\pi^\sharp : T^*M \longrightarrow TM, \quad \pi^\sharp(\alpha)(\beta) = \pi(\alpha, \beta).$$

Of course, T^*M should be considered together with the structure that allows us to treat it as a “generalized tangent bundle”, i.e. with its canonical structure of Lie algebroid: the anchor is π^\sharp , while the Lie bracket given by

$$(2) \quad [\alpha, \beta]_\pi = L_{\pi^\sharp(\alpha)}(\beta) - L_{\pi^\sharp(\beta)}(\alpha) - d\pi(\alpha, \beta), \quad (\forall) \alpha, \beta \in \Gamma(T^*M).$$

According to this philosophy, the analogue of the universal cover of a manifold should use “cotangent paths” instead of paths. Recall that a **cotangent path** in (M, π) is a path $a : [0, 1] \longrightarrow T^*M$, above some path $\gamma : [0, 1] \longrightarrow M$, such that

$$\pi^\sharp(a(t)) = \frac{d}{dt}\gamma(t).$$

Similarly, one can talk about cotangent homotopies and one defines the **Poisson homotopy groupoid** of (M, π) , denoted $\Sigma(M, \pi)$ (also called the Weinstein groupoid [7, 8]), as the space consisting of cotangent homotopy classes of paths:

$$\Sigma(M, \pi) = \frac{\text{cotangent paths}}{\text{cotangent homotopy}}.$$

The source/target maps $s, t : \Sigma(M, \pi) \longrightarrow M$ take a cotangent path into the initial/final point of the base path.

Definition 1.6. *The **Poisson homotopy bundle** of (M, π) at x is $P_x := s^{-1}(x)$ (the set of cotangent homotopy classes of cotangent paths starting at x).*

Recall that $\Sigma(M, \pi)$ is a groupoid, where the composition is given by concatenation of cotangent paths (here, to stay within the class of smooth paths, the concatenation is slightly perturbed using a bump function; however, up to cotangent homotopy, the result does not depend on the choice of the bump function - again, see [8] for details). In particular, $G_x := s^{-1}(x) \cap t^{-1}(x)$ is a group, which we will call the **Poisson homotopy group** of (M, π) at x . Also, the composition

defines a free action of G_x on P_x , and the quotient is identified with the symplectic leaf S_x through x , via the target map

$$P_x \longrightarrow S_x, \quad [a] \mapsto \gamma(1).$$

Regarding the smoothness of $\Sigma(M, \pi)$, one remarks that it is a quotient of the (Banach) manifold of cotangent paths of class C^1 . We are interested only in smooth structures which make the corresponding quotient map into a submersion. Of course, there is at most one such smooth structure on $\Sigma(M, \pi)$; when it exists, one says that $\Sigma(M, \pi)$ **is smooth**, or that (M, π) is **integrable**. Note that in this case $\Sigma(M, \pi)$ will be a finite dimensional manifold, but which may fail to be Hausdorff. If also the Hausdorffness condition is satisfied, we say that (M, π) is **Hausdorff integrable**. Completely analogously, one makes sense of the smoothness of P_x and of G_x . Note however that, whenever smooth, these two will be automatically Hausdorff. Moreover, the smoothness of P_x is equivalent to that of G_x and this is controlled by the **monodromy map** at x , which is a group homomorphism

$$(3) \quad \partial : \pi_2(S) \longrightarrow G(\mathfrak{g}_x)$$

into the 1-connected Lie group $G(\mathfrak{g}_x)$ integrating \mathfrak{g}_x . Intuitively, ∂ encodes the variation of symplectic areas, while in the smooth case, ∂ can also be identified with the boundary in the homotopy long exact sequence associated to $P_x \longrightarrow S_x$. From [8], we mention here:

Proposition 1.7. *The Poisson homotopy bundle P_x at x is smooth if and only if the image of ∂_x is a discrete subgroup of $G(\mathfrak{g}_x)$.*

In this case P_x is smooth principal G_x -bundle over S_x , the Lie algebra of G_x is \mathfrak{g}_x , $\pi_0(G_x) \cong \pi_1(S)$ and identity component G_x^0 is isomorphic to $G(\mathfrak{g}_x)/\text{Im}(\partial_x)$.

Coming back to our normal forms:

Definition 1.8. *Assuming that P_x is smooth, the **first order local model** of (M, π) around $S = S_x$ is defined as the local model (in the sense of subsection 1.4) associated to the Poisson homotopy bundle. The Poisson structure on the local model (well defined up to diffeomorphisms) is denoted $j_S^1\pi$ and is called **the first order approximation of π along S** .*

The fact that the Poisson homotopy bundle encodes the first jet of π along S will be explained in the next subsection; the fact that $j_S^1\pi$ deserves the name of first order approximation of π along S is explained in section 3 (subsection 3.2).

1.6. The Poisson homotopy bundle II: via its Atiyah sequence. In this subsection we present a slightly different point of view on the Poisson homotopy bundle P_x . The main remark is that the P_x is not visible right away as a smooth principal bundle, but through its infinitesimal data, i.e. an “abstract Atiyah sequence”.

This point of view has several advantages. For instance, it will allow us to see that, indeed, P_x encodes the first order jet of π along the symplectic leaf S_x . It also implies that the local model can be constructed without the smoothness assumption on P_x . Also, this approach to P_x does not really need the use of [7] (integrability of Lie algebroids); actually, (abstract) Atiyah sequences appeared outside the theory of Lie algebroids, as the infinitesimal counterparts of principal bundles. However, we will appeal to the language of Lie algebroids, as it simplifies the discussion. In particular, an **abstract Atiyah sequence** over a manifold S is simply a transitive Lie algebroid A over S , thought of as the exact sequence of Lie algebroids:

$$(4) \quad 0 \longrightarrow \text{Ker}(\rho) \longrightarrow A \xrightarrow{\rho} TS \longrightarrow 0,$$

where ρ is the anchor map of A . Any principal G -bundle $p : P \rightarrow S$ gives rise to such a sequence, known as **the Atiyah sequence** associated to P :

$$0 \rightarrow P \times_G \mathfrak{g}^* \rightarrow TP/G \xrightarrow{(dp)} TS \rightarrow 0.$$

Here, the Lie algebroid is $A(P) := TP/G$ and the bracket on $\Gamma(A(P))$ comes from the Lie bracket of G -invariant vector fields on P , via the identification

$$\Gamma(A(P)) = \mathfrak{X}(P)^G.$$

Given an abstract Atiyah sequence (4) over S , one says that it is **integrable** if there exists a principal G -bundle P (for some Lie group G) such that A is isomorphic to $A(P)$; one also says that P integrates (4). This notion was already considered in [1] without any reference to Lie algebroids. However, it is clear that this condition is equivalent to the integrability of A as a transitive Lie algebroid (see also [20, 21, 7]). In particular, as for Lie groups, if (4) is integrable, then there exists a unique (up to isomorphism) 1-connected principal bundle integrating it.

Remark 2. The fact that abstract Atiyah sequences are the infinitesimal counterparts of principal bundles also follows from the fact that transitive groupoids are essentially the same thing as principal bundles: any principal G -bundle $p : P \rightarrow S$ induces a transitive Lie groupoid over S - the quotient of the pair groupoid of P modulo the diagonal action of P (called the gauge groupoid of P); conversely, any transitive Lie groupoid \mathcal{G} over S arises in this way- just choose $x \in S$ and choose $t : P_x = s^{-1}(x) \rightarrow S$, with structural group the isotropy group $G_x = s^{-1}(x) \cap t^{-1}(x)$.

Back to our Poisson manifold (M, π) , one has an abstract Atiyah sequence

$$(5) \quad 0 \rightarrow \nu_S^* \rightarrow T_S^*M \xrightarrow{\pi^\sharp} TS \rightarrow 0.$$

above any symplectic leaf $S = S_x$. Of course, T_S^*M is just the restriction to S of the cotangent Lie algebroid (see the previous subsection). The description of P_x in terms of paths can also be seen as the general construction of [7] applied to this Lie algebroid. We conclude (using the above mentioned references):

Proposition 1.9. *Given $x \in S$, the Poisson homotopy bundle P_x is smooth if and only if the abstract Atiyah sequence (5) is integrable. Moreover, in this case P_x is the unique integration of (5) which is 1-connected.*

Next, we show that the abstract Atiyah sequence (5) (hence also the Poisson homotopy bundle) encodes the first jet of π along S . Consider the differential graded Lie algebra $\mathfrak{X}^\bullet(M)$ of multivector fields on M , the sub-algebra $\mathfrak{X}_S^\bullet(M)$ consisting of multivector fields whose restriction to S belongs to $\mathfrak{X}^\bullet(S)$ and the square $I^2(S) \subset C^\infty(M)$ of the ideal $I(S)$ of smooth functions which vanish on S . The first order jets along S are controlled by the map of graded Lie algebras

$$j_S^1 : \mathfrak{X}_S^\bullet(M) \rightarrow \mathfrak{X}_S^\bullet(M)/I^2(S)\mathfrak{X}^\bullet(M).$$

We see that, for the symplectic (S, ω_S) , first jets along S of Poisson structures that have (S, ω_S) as a symplectic leaf correspond to elements

$$\tau \in \mathfrak{X}_S^2(M)/I^2(S)\mathfrak{X}^2(M) \quad \text{satisfying} \quad [\tau, \tau] = 0$$

and with the property that the restriction map

$$r_S : \mathfrak{X}_S^2(M)/I^2(S)\mathfrak{X}^2(M) \rightarrow \mathfrak{X}^2(S), \quad \text{sends } \tau \text{ to } \omega_S^{-1}.$$

We denote by $J_{(S, \omega_S)}^1 \text{Poiss}(M)$ the set of such elements τ . It is interesting that any such τ comes from a Poisson structure π defined on a neighborhood of S in M (this

follows from the discussion below). Note that, starting with (S, ω_S) , one always has a short exact sequence of vector bundles over S :

$$(6) \quad 0 \longrightarrow \nu_S^* \longrightarrow T_S^* M \xrightarrow{\rho_{\omega_S}} TS \longrightarrow 0, \quad \text{with} \quad \rho_{\omega_S}(\xi) = \omega_S^{-1}(\xi|_{TS}).$$

Proposition 1.10. *Given a submanifold S of M and a symplectic form ω_S on S , there is a 1-1 correspondence between elements $\tau \in J_{(S, \omega_S)}^1 \text{Poiss}(M)$ and Lie brackets $[\cdot, \cdot]_\tau$ on $T_S^* M$ making (6) into an abstract Atiyah sequence.*

Remark 3. The local model (from our main theorem) can be described without using the smoothness P_x . This was explained by Vorobjev (see [27, 28] and our section 3). Here we indicate a different approach using Dirac structures. We start with a symplectic manifold (S, ω_S) and an abstract Atiyah sequence over S

$$0 \longrightarrow K \longrightarrow A \xrightarrow{\rho} TS \longrightarrow 0,$$

and we construct a Poisson structure around the zero section of the dual K^* of the vector bundle K , generalizing the integrable case (note that, if $A \cong A(P)$, then $K^* = P \times_G \mathfrak{g}^*$). Choose a splitting $\theta : A \longrightarrow K$. As for Lie algebras, one has a “fiberwise linear Poisson structure” π_{lin} on A^* . Using $\theta^* : K^* \longrightarrow A^*$, we can pull-back π_{lin} to a Dirac structure L_θ on K^* . Let $\omega \in \Omega^2(K^*)$ be the pull-back of ω_S along the projection. The gauge transform of L_θ of with respect to ω , $L(\theta) := L_\theta^\omega$, gives the local model in full generality (here, again, $S \subset \text{supp}(L(\theta))$). Indeed, one checks that, when A comes from a principal bundle P , this is precisely the Dirac structure mentioned in Remark 1); for full details, see [23].

1.7. More on integrability. There are a few more aspects of $\Sigma(M, \pi)$ that deserve to be recalled here. We use as references [7, 4, 8, 22, 24].

First of all, while $\Sigma(M, \pi)$ uses the cotangent Lie algebroid T^*M of the Poisson manifold, a similar construction applies to any Lie algebroid A . The outcome is a topological groupoid $\mathcal{G}(A)$ whose smoothness is equivalent to the integrability of A . For instance, when $A = \mathfrak{g}$ is a Lie algebra, $\mathcal{G}(\mathfrak{g})$ is the unique 1-connected Lie group with Lie algebra \mathfrak{g} . When $A = TM$, then $\mathcal{G}(TM)$ is the usual homotopy groupoid of M . Implicit in the previous discussion above is the fact that, for a symplectic leaf S of a Poisson manifold (M, π) , $\mathcal{G}(T^*M|_S) = \mathcal{G}(T^*M)|_S$ is encoded by the Poisson homotopy bundle (see Remark 2).

The second point is that, while $\mathcal{G}(A)$ makes sense for any Lie algebroid A , in the Poisson case, $\mathcal{G}(T^*M) = \Sigma(M, \pi)$ is a symplectic groupoid, i.e., comes endowed with a symplectic structure ω which is compatible with the groupoid composition (i.e. is multiplicative). There are two uniqueness phenomena here:

- ω is the unique multiplicative form for which $s : \Sigma \longrightarrow M$ is a Poisson map.
- For a general symplectic groupoid (Σ, ω) over a manifold M , there is a unique Poisson structure π on M such that $s : \Sigma \longrightarrow M$ is a Poisson map.

Combining with Lie II for Lie algebroids we deduce that, for a Poisson manifold (M, π) , if (Σ, ω) is a symplectic Lie groupoid with 1-connected s -fibers, and if $s : (\Sigma, \omega) \longrightarrow (M, \pi)$ is a Poisson map, then Σ is isomorphic to $\Sigma(M, \pi)$. For us, this gives a way of computing Poisson homotopy bundles more directly.

Example 1.11. Consider the linear Poisson structure $(\mathfrak{g}^*, \pi_{\text{lin}})$ on the dual of a Lie algebra \mathfrak{g} . Let $G = G(\mathfrak{g})$. We have that T^*G , endowed with the canonical symplectic structure, is the s -fiber 1-connected symplectic groupoid integrating π_{lin} . Using the identifying $T^*G = G \times \mathfrak{g}^*$, given by left translations, the groupoid structure is that of the action groupoid of G on \mathfrak{g}^* . It follows that, for any $\xi \in \mathfrak{g}^*$, the symplectic leaf through ξ is the coadjoint orbit \mathcal{O}_ξ and the associated Poisson homotopy bundle is precisely the G_ξ -bundle $G \longrightarrow \mathcal{O}_\xi$.

2. THE MAIN THEOREM AGAIN: REFORMULATIONS AND SOME EXAMPLES

In this section we give a complete statement the main theorem, two equivalent formulations and several examples.

As a summary of the previous section: given (M, π) and the symplectic leaf S through x , the first order jet of π along S is encoded in the Poisson homotopy bundle $P_x \rightarrow S$; out of it we produced the local model $P_x \times_{G_x} \mathfrak{g}_x^*$ which, around S , is a Poisson manifold admitting S as a symplectic leaf.

Main Theorem 1 (complete version). *Let (M, π) be a Poisson manifold, S a compact symplectic leaf and $x \in S$. If P_x , the Poisson homotopy bundle at x , is smooth, compact, with*

$$(7) \quad H^2(P_x; \mathbb{R}) = 0,$$

then there exists a Poisson diffeomorphism between an open neighborhood of S in M and a Poisson neighborhood of S in the local model $P_x \times_{G_x} \mathfrak{g}_x^$ associated to P_x , which is the identity on S .*

Remark 4. The open neighborhood of S in M can be chosen to be saturated. Indeed, by the comments following Definition 1.4, the open in the local model can be chosen of the type $P_x \times_{G_x} V$, and this is a union of compact symplectic leaves.

Comparing with the classical results from foliation theory and group actions, the surprising condition is (7). As we shall soon see, this condition is indeed necessary and it is related to integrability. However, as the next proposition shows, this condition is not needed in the Hausdorff integrable case.

Proposition 2.1. *In the main theorem, if S admits a neighborhood which is Hausdorff integrable, then the assumption (7) can be dropped.*

Note also that, in contrast with the proposition, if M is compact, then the conditions of our main theorem cannot hold *at all* points $x \in M$ (since it would follow that $\Sigma(M, \pi)$ is compact and its symplectic form is exact- see [9]).

Next, since the conditions of the theorem may be difficult to check in explicit examples, we reformulate them in terms of the Poisson homotopy group G_x at x .

Proposition 2.2. *The conditions of the main theorem are equivalent to:*

- (1) *The leaf S is compact.*
- (2) *The Poisson homotopy group G_x is smooth and compact.*
- (3) *The dimension of the center of G_x equals to the rank of $\pi_2(S, x)$.*

Proof. We already know that the smoothness of P_x is equivalent to that of G_x while, under this smoothness condition, compactness of P_x is clearly equivalent to that of S and G_x . Hence, assuming (1) and (2), we still have to show that (3) is equivalent to $H^2(P_x) = 0$. Since G_x is compact, \mathfrak{g}_x is a product of a semi-simple Lie algebra \mathfrak{h} of compact type with its center ζ . Therefore $G(\mathfrak{g}_x) = H \times \zeta$, with H compact 1-connected, and ∂_x takes values in $Z \times \zeta$, where $Z = Z(H)$ is a finite group. Since $\pi_2(P_x)$ can be identified with $\text{Ker}(\partial_x)$ we have an exact sequence

$$0 \rightarrow \pi_2(P_x) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \pi_2(S) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\partial_{\mathbb{R}}} \zeta$$

where, since P_x is 1-connected, the first term is canonically isomorphic to $H_2(P_x; \mathbb{R})$. Finally, since the connected component of the identity in G_x is $(H \times \zeta)/\text{Im}(\partial_x)$, its compactness implies that $\partial_{\mathbb{R}}$ is surjective. Hence a short exact sequence

$$0 \rightarrow H_2(P_x) \rightarrow \pi_2(S) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\partial_{\mathbb{R}}} \zeta \rightarrow 0.$$

Therefore, condition (3) is equivalent to the vanishing of $H_2(P_x)$. □

Next, using the monodromy group, one can also get rid of the G_x . Recall [8, 7] that the monodromy group of (M, π) at x , denoted by \mathcal{N}_x , is the image of ∂_x (see (3)) intersected with the connected component of the center of $G(\mathfrak{g}_x)$. Using the exponential it can be viewed as a subgroup of the center of \mathfrak{g}_x

$$\mathcal{N}_x \subset Z(\mathfrak{g}_x).$$

Proposition 2.3. *The conditions of the main theorem are equivalent to:*

- (1) *The leaf S is compact with finite fundamental group.*
- (2) *The isotropy Lie algebra \mathfrak{g}_x is of compact type.*
- (3) *\mathcal{N}_x is a lattice in $Z(\mathfrak{g}_x)$.*
- (4) *The dimension of $Z(\mathfrak{g}_x)$ equals the rank of $\pi_2(S, x)$.*

Proof. Let $\zeta = Z(\mathfrak{g}_x)$ and denote by $\tilde{\mathcal{N}}_x$ the image of ∂_x . The discreteness of \mathcal{N}_x is equivalent to that of $\tilde{\mathcal{N}}_x$ [8], hence to the smoothness of G_x . The compactness of G_x is equivalent to the following two conditions: $\pi_0(G_x)$ is finite and the connected component of the identity G_x° is compact. From Proposition 1.7, the first condition is equivalent to $\pi_1(S)$ -finite, while the second one to $(Z \times \zeta)/\tilde{\mathcal{N}}_x$ being compact. The last condition is equivalent to ζ/\mathcal{N}_x being compact (hence to \mathcal{N}_x being a lattice in ζ). For this, note that ζ/\mathcal{N}_x injects naturally into $(Z \times \zeta)/\tilde{\mathcal{N}}_x$ and there is a surjection of $Z \times \zeta/\mathcal{N}_x$ onto $(Z \times \zeta)/\tilde{\mathcal{N}}_x$. Condition (4) is equivalent to (3) from Proposition 2.2. \square

Example 2.4. We now give an example in which all conditions of the theorem are satisfied, except for the vanishing of $H^2(P_x)$, and in which the conclusion of the theorem fails. Consider the 2-sphere S^2 , with coordinates denoted (u, v, w) , endowed with the Poisson structure π_{S^2} which is the inverse of the area form

$$\omega_{S^2} = (udv \wedge dw + vdw \wedge du + wdu \wedge dv).$$

Consider also the linear Poisson structure on $so(3)^* \cong \mathbb{R}^3 = \{(x, y, z)\}$:

$$\pi_{\text{lin}} = x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Its symplectic leaves are the spheres of radius $r > 0$, S_r^2 , with the symplectic form

$$\omega_r = \frac{1}{r^2}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy),$$

and the origin. Finally, let (M, π_0) be the product of these two Poisson manifolds

$$M = S^2 \times \mathbb{R}^3, \quad \pi_0 = \pi_{S^2} + \pi_{\text{lin}}.$$

The symplectic leaves are: $(S, \omega_S) := (S^2 \times \{0\}, \omega_{S^2})$ and, for $r > 0$, $(S^2 \times S_r^2, \omega_{S^2} + \omega_r)$. The abstract Atiyah sequence above S is the product of the Lie algebroids TS and $so(3)$. Hence, for $x \in S$, G_x equals to $G(so(3)) = \text{Spin}(3)$ and the Poisson homotopy bundle is $P_x = S^2 \times \text{Spin}(3) \cong S^2 \times S^3$. Using the trivial connection on P_x , one finds that (M, π_0) coincides with the resulting local model. Note that all the conditions of the theorem are satisfied, except for the vanishing of $H^2(P_x)$.

Let us now modify π_0 without modifying $j_S^1 \pi_0$; we consider

$$\pi = (1 + r^2)\pi_{S^2} + \pi_{\text{lin}},$$

Note that π has the same leaves as π_0 , but with different symplectic forms:

$$(S, \omega_S), \quad (S^2 \times S_r^2, \frac{1}{1 + r^2}\omega_{S^2} + \omega_r), \quad r > 0.$$

We claim that π is not Poisson diffeomorphic, around S , to π_0 . Assume it is. Then, for any r small enough, we find r' and a symplectomorphism

$$\phi : (S^2 \times S_r^2, \frac{1}{1 + r^2}\omega_{S^2} + \omega_r) \longrightarrow (S^2 \times S_{r'}^2, \omega_{S^2} + \omega_{r'}).$$

Comparing the symplectic volumes, we find $r' = r/(1 + r^2)$. On the other hand, ϕ sends the first generator γ_1 of $\pi_2(S^2 \times S_r^2)$ into a combination $m\gamma_1 + n\gamma_2$ with m and n integers. Computing the symplectic areas of these elements, we obtain:

$$\int_{\gamma_1} \left(\frac{1}{1 + r^2} \omega_{S^2} + \omega_r \right) = \int_{m\gamma_1 + n\gamma_2} (\omega_{S^2} + \omega_{r'}),$$

thus $1/(1 + r^2) = m + nr' = m + nr/(1 + r^2)$. This cannot be satisfied for all r (even small enough), because it forces r to be an algebraic number.

Also, the monodromy group of the leaf $S^2 \times S_r^2$ is the subgroup of \mathbb{R} generated by 4π and $4\pi(-2r)/(1 + r^2)^2$; therefore, if it is discrete, then again r is algebraic. This shows that π is not integrable on any open neighborhood of S .

Example 2.5 (the regular case). Assume that (M, π) is regular and fix a symplectic leaf S . Then the resulting normal form is a refinement of local Reeb stability: the model for the underlying foliation is the same, while our theorem also specifies the leafwise symplectic forms. To see this, let

$$\Gamma = \pi_1(S, x), \quad \nu_x = T_x M / T_x S$$

In this case $\mathfrak{g}_x = \nu_x^*$ is abelian and Proposition 1.7 gives a short exact sequence

$$G_x^\circ \longrightarrow G_x \longrightarrow \Gamma.$$

Hence G_x° is abelian, the Poisson homotopy bundle is a principal G_x° -bundle over the universal cover \tilde{S} ; in conclusion, as a foliated manifold, the local model is

$$P_x \times_{G_x} \nu_x \cong \tilde{S} \times_\Gamma \nu_x.$$

The Poisson structure comes from a family of symplectic forms ω_t on \tilde{S} of the form:

$$\omega_t = p^*(\omega_S) + t_1\omega_1 + \dots + t_q\omega_q, \quad t = (t_1, \dots, t_q) \in \nu_x,$$

where we have chosen a basis of ν_x , $p : \tilde{S} \rightarrow S$ is the projection and $\omega_i \in \Omega^2(\tilde{S})$ are representatives of the components of the monodromy map $\partial : \pi_2(S) \rightarrow \nu_x^*$, i.e.

$$\partial(\sigma) = \left(\int_\sigma \omega_1, \dots, \int_\sigma \omega_q \right), \quad [\sigma] \in \pi_2(S, x).$$

Note that this condition determines uniquely $[\omega_i] \in H^2(\tilde{S})$. Since the rank of $\pi_2(S)$ equals the $b_2(\tilde{S})$, by Proposition 2.3, the assumptions of the theorem become: \tilde{S} is compact (so local Reeb stability applies) and $[\omega_1], \dots, [\omega_q]$ is a basis of $H^2(\tilde{S})$.

Also the description of the monodromy map simplifies as shown in [8]. Let σ be a 2-sphere in S , with north pole P_N at x and $t \in \nu_x$. Consider a smooth family σ_ϵ , defined for ϵ small, of leafwise 2-spheres such that $\sigma_0 = \sigma$ and the vector $\dot{\sigma}_0(P_N)$ represents t . Then the monodromy map on σ is:

$$(8) \quad \partial(\sigma)(t) = \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{\sigma_\epsilon} \omega_\epsilon.$$

For instance, when S is simply connected, then by Reeb stability $M = S \times \mathbb{R}^q$ with the trivial foliation so that the Poisson structure on M is determined by a family $\{\omega_t\}$ of symplectic forms with $\omega_0 = \omega_S$. Then ω_k is simply

$$\omega_k = \frac{\partial}{\partial t_k} \Big|_{t=0} \omega_t.$$

This class of examples reveals again the necessity of the condition (7). In order to see this, observe that the other conditions of the theorem are equivalent to the first three conditions of Proposition 2.3, whereas the first two are automatically satisfied in this case. By the third condition, that \mathcal{N}_x is a lattice in ν_x^* , we can choose a basis of ν_x^* which is also a \mathbb{Z} -basis of \mathcal{N}_x . As a result, the $[\omega_i]$'s belong to $H^2(S; \mathbb{Z})$ and are linearly independent. If the fourth condition is not satisfied,

we can find a closed 2-form λ , with $[\lambda] \in H^2(S; \mathbb{Z})$ and which is not in the span of $[\omega_1], \dots, [\omega_q]$. The Poisson structure corresponding to the family of 2-forms

$$\omega_t := \omega_S + t_1 \omega_1 + \dots + t_q \omega_q + t_1^2 \lambda$$

satisfies all the conditions of the theorem except for (7), but it is not linearizable around S . Otherwise, we could find a diffeomorphism of the form $(x, t) \mapsto (\phi_t(x), \tau(t))$ with $\tau(0) = 0$, $\phi_0(x) = x$ and such that

$$\phi_t^* \omega_S + \sum \tau_i(t) \phi_t^* \omega_i = \omega_S + \sum t_i \omega_i + t_1^2 \lambda.$$

Since ϕ_t^* is the identity in cohomology, we get a contradiction.

Related to our Proposition 2.1, we see that this Poisson structure is not integrable on any open neighborhood of S . This follows by computing the monodromy groups using (8) at (t_1, \dots, t_q) ; they are discrete if and only if $t_1 \in \mathbb{Q}$.

Example 2.6 (Duistermaat-Heckman variation formula). Next, we indicate the relationship of our theorem with the theorem of Duistermaat and Heckman from [13]. We first recall their result. Let (M, ω) be a symplectic manifold endowed with a Hamiltonian action of a torus T with proper moment map $J : M \rightarrow \mathfrak{t}^*$ and let $\xi_0 \in \mathfrak{t}^*$ be a regular value of J . Assume that the action of T on $J^{-1}(\xi_0)$ is free. Let U be a ball around ξ_0 consisting of regular values of J . The symplectic quotients

$$S_\xi := J^{-1}(\xi)/T$$

come with symplectic forms denoted σ_ξ , $\xi \in U$. There are canonical isomorphisms

$$H^2(S_\xi) \cong H^2(S_{\xi_0})$$

for $\xi \in U$ and the Duistermaat-Heckman theorem asserts that, in cohomology,

$$[\sigma_\xi] = [\sigma_{\xi_0}] + \langle c, \xi - \xi_0 \rangle,$$

where c is the Chern class of the T -bundle $J^{-1}(\xi_0) \rightarrow S_{\xi_0}$. This is related to our theorem applied to the Poisson manifold N/T , where $N := J^{-1}(U)$. The symplectic leaves of N/T are precisely the S_ξ 's and N/T is integrable by the symplectic groupoid (see Proposition 4.6 in [16])

$$\mathcal{G} := (N \times_J N)/T \rightrightarrows N/T,$$

with symplectic structure induced by $pr_1^*(\omega) - pr_2^*(\omega) \in \Omega^2(N \times N)$. The isotropy groups of \mathcal{G} are all isomorphic to T and the s -fibers are isomorphic (as principal T -bundles) to the fibers of J . As shown in [13], if U is small enough all fibers of J are diffeomorphic as T -bundles, so if we are assuming that $J^{-1}(\xi_0)$ is 1-connected, then \mathcal{G} is the 1-connected symplectic groupoid integrating N/T . In particular the Poisson homotopy bundle corresponding to S_{ξ_0} is the T -bundle $J^{-1}(\xi_0) \rightarrow S_{\xi_0}$, hence the Chern class c is the same appearing in the construction of the local model (see Example 1.5) and is also the monodromy map (8), interpreted as a cohomology class with coefficients in $\nu_x = \mathfrak{t}$. In this case the condition $H^2(J^{-1}(\xi_0)) = 0$ is not required, since we can apply directly Proposition 2.1.

Example 2.7. Consider $(\mathfrak{g}^*, \pi_{\text{lin}})$, the dual of a Lie algebra \mathfrak{g} , endowed with the linear Poisson structure. Let $\xi \in \mathfrak{g}^*$. As we have seen in Example 1.11, the leaf through ξ is the coadjoint orbit \mathcal{O}_ξ and the corresponding Poisson homotopy bundle is $P_\xi = G$, the 1-connected Lie group integrating \mathfrak{g} . So the hypothesis of our theorem reduces to the condition that \mathfrak{g} is semi-simple of compact type. Note also that the resulting local form around \mathcal{O}_ξ implies the linearizability of the transversal Poisson structure [30] to \mathcal{O}_ξ , which fails for general Lie algebras [30]-Errata.

Of course, one may wonder about a direct argument. This is possible- and actually reveals a slightly weaker hypothesis: one needs that \mathcal{O}_ξ is an embedded submanifold and that $\xi \in \mathfrak{g}$ is split in the sense that there is a G_ξ -invariant projection

$p : \mathfrak{g} \longrightarrow \mathfrak{g}_\xi$. For the details, we use [17] and [26]. Observe that the normal bundle of \mathcal{O}_ξ is isomorphic to $G \times_{G_\xi} \mathfrak{g}_\xi^*$. The projection p induces a tubular neighborhood:

$$\varphi : G \times_{G_\xi} \mathfrak{g}_\xi^* \rightarrow \mathfrak{g}^*, \quad [g, \eta] \rightarrow \text{Ad}_{g^{-1}}^*(\xi + p^*(\eta)),$$

and a G -invariant principle connection on $G \longrightarrow \mathcal{O}_\xi$:

$$\theta \in \Omega^1(G; \mathfrak{g}_\xi), \quad \theta_g = l_{g^{-1}}^*(p).$$

Let $\Omega \in \Omega^2(G \times \mathfrak{g}_\xi^*)$ be the resulting 2-form (see subsection 1.4). The nondegeneracy locus of Ω can be described more explicitly. Let $\mathcal{N} \subset \mathfrak{g}_\xi^*$ be the set of points $\eta \in \mathfrak{g}_\xi^*$, for which the coadjoint orbit through $\xi + p^*(\eta)$ and the affine space $\xi + p^*(\mathfrak{g}_\xi^*)$ are transversal at $\xi + p^*(\eta)$. section 2.3.1 and Theorem 2.3.7 in [17] show that:

- $G \times \mathcal{N}$ is the open in $G \times \mathfrak{g}_\xi^*$ on which Ω is nondegenerate,
- $G \times_{G_\xi} \mathcal{N}$ is the open in $G \times_{G_\xi} \mathfrak{g}_\xi^*$ on which the differential of φ is invertible.

Let π_ξ denote the Poisson structure on $G \times_{G_\xi} \mathcal{N}$ obtained by reduction. Theorem 1 from section 1.3 in [26] shows that φ is a Poisson map

$$\varphi : (G \times_{G_\xi} \mathcal{N}, \pi_\xi) \longrightarrow (\mathfrak{g}^*, \pi_{\text{lin}}).$$

We deduce that, around the embedded \mathcal{O}_ξ , φ provides a Poisson diffeomorphism.

3. POISSON STRUCTURES AROUND A SYMPLECTIC LEAF: THE ALGEBRAIC FRAMEWORK

In this section we discuss the algebraic framework which encodes the behavior of Poisson structures around symplectic leaves (an improvement of the framework of [10]). This will allow us to regard the local model as a first order approximation and to produce smooth paths of Poisson bivectors (to be used in the next section).

Since we are interested in the local behavior of Poisson structures around an embedded symplectic leaf, we may restrict our attention to a tubular neighborhood. Throughout this section $p : E \longrightarrow S$ is a vector bundle over a manifold S . Consider the vertical sub-bundle and the space of vertical multi-vector fields on E denoted

$$V := \ker(dp) \subset TE, \quad \mathfrak{X}_V^\bullet(E) = \Gamma(\wedge^\bullet V) \subset \mathfrak{X}^\bullet(E).$$

We have that $\mathfrak{X}_V^\bullet(E)$ is a graded Lie sub-algebra of the space $\mathfrak{X}^\bullet(E)$ of multivector fields on E , with respect to the Schouten bracket. The Lie algebra grading is:

$$\deg(X) := |X| - 1 = q - 1 \quad \text{for } X \in \mathfrak{X}^q(E).$$

Given a vector bundle F over S , denote the space of F -valued forms on S by:

$$\Omega^\bullet(S, F) := \Gamma(\wedge^\bullet T^*S \otimes F) = \Omega^\bullet(S) \otimes_{C^\infty(S)} \Gamma(F).$$

More generally, for any $C^\infty(S)$ -module \mathfrak{X} , denote by

$$\Omega^\bullet(S, \mathfrak{X}) := \Omega^\bullet(S) \otimes_{C^\infty(S)} \mathfrak{X},$$

the space of antisymmetric forms on S with values in \mathfrak{X} .

3.1. The graded Lie algebra $(\tilde{\Omega}_E, [\cdot, \cdot]_\times)$ and horizontally non-degenerate Poisson structures. We first recall the graded Lie algebra Ω_E of [10]. We introduce Ω_E as the bi-graded vector space whose elements of bi-degree (p, q) are p -forms on S with values in the $C^\infty(S)$ -module $\mathfrak{X}_V^q(E)$:

$$\Omega_E^{p,q} = \Omega^p(S, \mathfrak{X}_V^q(E)).$$

The \mathbb{Z} -grading is $\deg = p + q - 1$ on $\Omega_E^{p,q}$, and the bracket is

$$[\varphi \otimes X, \psi \otimes Y] = (-1)^{|\psi|(|X|-1)} \varphi \wedge \psi \otimes [X, Y].$$

We will need an enlargement $\tilde{\Omega}_E$ of Ω_E . As a bi-graded vector space, it is

$$\tilde{\Omega}_E = \Omega^\bullet(S, \mathfrak{X}_V^\bullet(E)) + \Omega^\bullet(S, \mathfrak{X}_P(E)) \subset \Omega^\bullet(S, \mathfrak{X}^\bullet(E)),$$

where $\mathfrak{X}_P(E)$ is the space of **projectable vector fields** on E , i.e. vector fields $X \in \mathfrak{X}(E)$ with the property that there is a vector field on S , denoted $p_S(X) \in \mathfrak{X}(S)$, such that $dp(X) = p_S(X)$. Hence, in bi-degree (p, q) we have

$$\tilde{\Omega}_E^{p,q} = \begin{cases} \Omega^p(S, \mathfrak{X}_V^q(E)) & \text{if } q \neq 1 \\ \Omega^p(S, \mathfrak{X}_P(E)) & \text{if } q = 1 \end{cases}.$$

The relationship between $\tilde{\Omega}_E$ and Ω_E is similar to the one between the Lie algebras $\mathfrak{X}_V(E)$ and $\mathfrak{X}_P(E)$: we have a short exact sequence of vector spaces:

$$0 \longrightarrow \Omega_E \longrightarrow \tilde{\Omega}_E \xrightarrow{p_S} \Omega(S, TS) \longrightarrow 0.$$

Next, we show that this is naturally a sequence of graded Lie algebras. On $\Omega(S, TS)$ we consider the **Fröhlicher-Nijenhuis-Bracket**, denoted $[\cdot, \cdot]_F$, which we recall using section 13 of [19]. The key-point is that $\Omega(S, TS)$ can be identified with the space of derivations of the graded algebra $\Omega(S)$, which commute with the DeRham differential, and, as a space of derivations, it inherits a natural Lie bracket. In more detail, for $u = \alpha \otimes X \in \Omega^r(S, TS)$, the operator $\mathcal{L}_u := [i_u, d]$ on $\Omega(S)$ is given by:

$$\mathcal{L}_u(\omega) = \alpha \wedge \mathcal{L}_X(\omega) + (-1)^r d\alpha \wedge i_X(\omega).$$

The resulting commutator bracket on $\Omega(S, TS)$ is:

$$[u, v]_F = \mathcal{L}_u(\beta) \otimes Y - (-1)^{rs} \mathcal{L}_v(\alpha) \otimes X + \alpha \wedge \beta \otimes [X, Y]$$

for $u = \alpha \otimes X \in \Omega^r(S, TS)$, $v = \beta \otimes Y \in \Omega^s(S, TS)$. With these $(\Omega(S, TS), [\cdot, \cdot]_F)$ is a graded Lie algebra, where the grading $\deg = r$ on $\Omega^r(S, TS)$. Consider the element corresponding to the identity map of TS , denoted by:

$$\gamma_S \in \Omega^1(S, TS)$$

Then γ_S is central in $(\Omega(S, TS), [\cdot, \cdot]_F)$ and it represents the DeRham differential:

$$(9) \quad \mathcal{L}_{\gamma_S} = d : \Omega^\bullet(S) \longrightarrow \Omega^{\bullet+1}(S).$$

Next, the operations involving $\Omega(S, TS)$ have the following lifts to E :

- With the short exact sequence

$$0 \longrightarrow \Omega(S, \mathfrak{X}_V(E)) \longrightarrow \Omega(S, \mathfrak{X}_P(E)) \xrightarrow{p_S} \Omega(S, TS) \longrightarrow 0$$

in mind, there is a natural lift of $[\cdot, \cdot]_F$ to the middle term, which we denote by the same symbol. Actually, realizing

$$\Omega(S, \mathfrak{X}_P(E)) \xrightarrow{p^*} \Omega(E, TE),$$

this is just the restriction of the Fröhlicher-Nijenhuis-Bracket on $\Omega(E, TE)$.

- The action \mathcal{L} of $\Omega(S, TS)$ on $\Omega(S)$ induces an action of $\Omega(S, \mathfrak{X}_P(E))$ on Ω_E , for $u = \alpha \otimes X \in \Omega(S, \mathfrak{X}_P(E))$ and $v = \omega \otimes Y \in \Omega_E$, we have:

$$\mathcal{L}_u(v) = \mathcal{L}_{p_S(u)}(\omega) \otimes Y + \alpha \wedge \omega \otimes [X, Y].$$

Putting everything together, the following is straightforward:

Proposition 3.1. *The following bracket defines a graded Lie algebra on $\tilde{\Omega}_E$:*

$$[u, v]_\kappa = \begin{cases} [u, v] & \text{for } u, v \in \Omega_E, \\ \mathcal{L}_u(v) & \text{for } u \in \Omega(S, \mathfrak{X}_P(E)), v \in \Omega_E, \\ [u, v]_F & \text{for } u, v \in \Omega(S, \mathfrak{X}_P(E)), \end{cases}$$

Moreover, we have a short exact sequence of graded Lie algebras:

$$0 \rightarrow (\Omega_E^\bullet, [\cdot, \cdot]) \rightarrow (\tilde{\Omega}_E^\bullet, [\cdot, \cdot]_\kappa) \xrightarrow{p_S} (\Omega^\bullet(S, TS), [\cdot, \cdot]_F) \rightarrow 0.$$

We encode a bit more of the structure of the algebra $(\tilde{\Omega}_E, [\cdot, \cdot]_\kappa)$ in the following lemma, whose proof is also straightforward:

Lemma 3.2. *Identifying $\Omega(S) \cong p^*(\Omega(S)) \subset \tilde{\Omega}_E$, $\Omega(S)$ is a central ideal in Ω_E . The induced representation of $\tilde{\Omega}_E$ on $\Omega(S)$ factors through p_S , i.e.*

$$[u, \omega]_\times = \mathcal{L}_u(\omega) = \mathcal{L}_{p_S(u)}(\omega), \quad (\forall) u \in \tilde{\Omega}_E, \omega \in \Omega(S).$$

As an illustration of the use of $\tilde{\Omega}_E$, we look at **Ehresmann connections** on E . Viewing such a connection as a $C^\infty(S)$ -linear map which associates to a vector field X on S its horizontal lift to E , we see that it is the same thing as an element in $\Gamma \in \tilde{\Omega}_E^{1,1}$ which satisfies $p_S(\Gamma) = \gamma_S$. Also the curvature R_Γ of Γ is just $R_\Gamma = \frac{1}{2}[\Gamma, \Gamma]_\times \in \Omega_E^{2,1}$.

We introduce the following generalization of flat Ehresmann connections (the terminology will be explained in Remark 5 below).

Definition 3.3. *A **Dirac element** on $p : E \rightarrow S$ is an element $\gamma \in \tilde{\Omega}_E^2$, satisfying*

$$[\gamma, \gamma]_\times = 0, \quad p_S(\gamma) = \gamma_S.$$

We use the following notations for the components of γ :

- γ^\vee for the $(2, 0)$ component- an element in $\mathfrak{X}_V^2(E)$.
- Γ_γ for the $(1, 1)$ component- an Ehresmann connection on E .
- \mathbb{F}_γ for the $(0, 2)$ component- an element in $\Omega^2(S, C^\infty(E)) = \Gamma(p^*\Lambda^2 T^*S)$.

*The **Poisson support** of γ is the set of points $e \in E$ at which \mathbb{F}_γ is non-degenerate.*

The relevance of such elements to the study of Poisson structures around a symplectic leaf comes from the fact that, while E plays the role of small tubular neighborhoods, on such E 's the following condition will be satisfied.

Definition 3.4. *A bivector $\theta \in \mathfrak{X}^2(E)$ is called **horizontally non-degenerate** if*

$$V_e + \theta^\sharp(V_e^\circ) = T_e E, \quad (\forall) e \in E,$$

where $V_e^\circ \subset T_e^ E$ is the annihilator of V_e in $T_e^* E$.*

Moreover, Vorobjev's Theorem 2.1 in [27] can be summarized in the following:

Proposition 3.5. *There is a 1-1 correspondence between*

- (1) *Dirac elements $\gamma \in \tilde{\Omega}_E^2$ with support equal to E .*
- (2) *Horizontally non-degenerate Poisson structures θ on E .*

The explicit construction of the 1-1 correspondence is important as well; we recall it below. The main point of our proposition is that the list of complicated equations from [27] takes now the compact form $[\gamma, \gamma]_\times = 0$. So, let θ be a horizontally non-degenerate Poisson structure on E . The non-degeneracy implies that

$$H_\theta = \theta^\sharp(V^\circ)$$

gives an Ehresmann connection on E ; we denote it by $\Gamma_\theta \in \tilde{\Omega}_E^{1,1}$. With respect to the resulting decomposition of TE , the mixed component of θ vanishes, i.e.

$$\theta = \theta^\vee + \theta^h \in \Lambda^2 V \oplus \Lambda^2 H_\theta.$$

The first term is the desired $(2, 0)$ -component. The second one is non-degenerate, thus, after passing from H_θ to TS and then taking the inverse, we get an element

$$\mathbb{F}_\theta \in \Gamma(p^*\Lambda^2 T^*S).$$

This will be the desired $(0, 2)$ -component. Explicitly,

$$(10) \quad \mathbb{F}_\theta(dp_e(\theta^{h\sharp}\eta), dp_e(\theta^{h\sharp}\mu)) = -\theta^h(\eta, \mu), \quad (\forall) \eta, \mu \in p^*(T^*S)_e.$$

Altogether, the 1-1 correspondence associates to θ the element

$$\gamma = \theta^\vee + \Gamma_\theta + \mathbb{F}_\theta \in \tilde{\Omega}_E^2$$

with \mathbb{F}_θ non-degenerate at all points of E and $p_S(\gamma) = p_S(\Gamma_\theta) = \gamma_S$.

Proof. (of the proposition) Conversely, it is clear that we can reconstruct θ from γ . One still has to check that $[\theta, \theta] = 0$ is equivalent to the equation $[\gamma, \gamma]_{\times} = 0$. By Vorobjev's formulas (Theorem 2.1 in [27]) and their interpretation using Ω_E from [10] (Theorem 4.2), the Poisson equation is equivalent to:

$$[\theta^{\vee}, \theta^{\vee}] = 0, [\Gamma_{\theta}, \theta^{\vee}]_{\times} = 0, R_{\Gamma_{\theta}} + [\theta^{\vee}, \mathbb{F}_{\theta}] = 0, [\Gamma_{\theta}, \mathbb{F}_{\theta}]_{\times} = 0.$$

Here we have used the remark that the covariant exterior derivative (denoted in [10] by $d_{\Gamma_{\theta}}, \partial_{\Gamma_{\theta}}$ respectively) can be given by $ad_{\Gamma_{\theta}} = [\Gamma_{\theta}, \cdot]_{\times} : \Omega_E \rightarrow \Omega_E$. Finally,

$$\begin{aligned} 0 = [\gamma, \gamma]_{\times} &= ([\theta^{\vee}, \theta^{\vee}]) + 2([\Gamma_{\theta}, \theta^{\vee}]_{\times}) + 2(R_{\Gamma_{\theta}} + [\theta^{\vee}, \mathbb{F}_{\theta}]) + 2([\Gamma_{\theta}, \mathbb{F}_{\theta}]_{\times}) \in \\ &\in \Omega_E^{0,3} \oplus \Omega_E^{1,2} \oplus \Omega_E^{2,1} \oplus \Omega_E^{3,0} = \Omega_E^3. \end{aligned}$$

□

Remark 5. Generalizing the case of Poisson structures, recall [2, 29] that a Dirac structure $L \subset TE \oplus T^*E$ is called horizontally non-degenerate if $L \cap (V \oplus V^{\circ}) = \{0\}$. The previous discussion applies with minor changes to such structures. The integrability condition (the four equations above) have been extended to horizontally non-degenerate Dirac structures (Corollary 2.8 in [2] and Theorem 2.9 in [29]). We find out that there is a 1-1 correspondence between

- Dirac elements $\gamma \in \tilde{\Omega}_E^2$.
- Horizontally non-degenerate Dirac structures on E .

Moreover, in this correspondence, the support of L (cf. Remark 1) coincides with the Poisson support of γ . Explicitly, the Dirac structure corresponding to γ is

$$L_{\gamma} = \text{Graph}(\gamma^{\vee\sharp} : H^{\circ} \rightarrow V) \oplus \text{Graph}(\mathbb{F}_{\gamma}^{\sharp} : H \rightarrow V^{\circ}),$$

where we use the decomposition $TE = V \oplus H$ induced by the connection Γ_{γ} .

Finally, we identify the Poisson cohomology complex $(\mathfrak{X}^{\bullet}(E), d_{\theta})$ ($d_{\theta} = [\theta, \cdot]$).

Proposition 3.6. *Let θ be a horizontally non-degenerate Poisson structure on E with corresponding Dirac element γ . Then there is an isomorphism of complexes*

$$\tau_{\theta} : (\mathfrak{X}^{\bullet}(E), d_{\theta}) \longrightarrow (\Omega_E^{\bullet}, ad_{\gamma}), \quad \text{where } ad_{\gamma} = [\gamma, \cdot]_{\times}.$$

Again, this is a reformulation of a result of [10], namely of Proposition 4.3, with the remark that the operator d_{θ} in *loc.cit.* is simply our ad_{γ} . For later use, we also give the explicit description of τ_{θ} . Identifying $\Omega_E = \Gamma(\wedge(p^*T^*S \oplus V))$,

$$(11) \quad \tau_{\theta} = \wedge^{\bullet} f_{\theta,*} : \mathfrak{X}^{\bullet}(E) \longrightarrow \Omega_E^{\bullet},$$

where f_{θ} is the bundle isomorphism

$$f_{\theta} := (-\mathbb{F}_{\theta}^{\sharp}, id_V) : H_{\theta} \oplus V = TE \longrightarrow p^*T^*S \oplus V.$$

3.2. The dilatation operators and jets along S . For $t \in \mathbb{R}, t \neq 0$ let $m_t : E \longrightarrow E$ be the fiberwise multiplication by t . Pull-back by m_t induces an automorphism

$$m_t^* : (\tilde{\Omega}_E, [\cdot, \cdot]) \longrightarrow (\tilde{\Omega}_E, [\cdot, \cdot]).$$

It preserves Ω_E and acts as the identity on $\Omega(S)$. Define the **dilation operators**:

$$\varphi_t : \tilde{\Omega}_E \longrightarrow \tilde{\Omega}_E, \quad \varphi_t(u) = t^{q-1} m_t^*(u), \quad \text{for } u \in \tilde{\Omega}_E^{(\bullet, q)}.$$

Remark 6. It is useful to describe this operation in local coordinates. Choose (x^i) coordinates for S and (y^a) linear coordinates on the fibers of E . An arbitrary element in $\Omega^p(S, \mathfrak{X}^q(E))$ is a sum of elements of type

$$a(x, y) dx^I \otimes \partial_{x^J} \wedge \partial_{y^K}$$

where I, J and K are multi-indices with $|I| = p, |J| + |K| = q$ and $a = a(x, y)$ is a smooth function. Such an element is in Ω_E if and only if it only contains terms

with $|J| = 0$. The elements in $\tilde{\Omega}_E$ are also allowed to contain terms with $|J| = 1$, but those terms must have $|K| = 0$, and the coefficient a only depending on x . Applying φ_t to such an element we find

$$t^{|J|-1}a(x, ty)dx^I \otimes \partial_{x^J} \wedge \partial_{y^K}.$$

Lemma 3.7. φ_t preserves the bi-degree, is an automorphism of the graded Lie algebra $\tilde{\Omega}_E$ and preserves Ω_E .

Proof. Due to its functoriality, m_t^* has similar properties. Due to the shift degree by 1 in the Lie degree, also the multiplication by t^{q-1} has the same properties. Hence also the composition of the two operations, i.e. φ_t , has the desired properties. \square

Together with φ_t we also introduce the following subspaces of $\tilde{\Omega}_E$ for $l \in \mathbb{Z}$:

$$\begin{aligned} \text{gr}_l(\tilde{\Omega}_E) &= \{u \in \tilde{\Omega}_E : \varphi_t(u) = t^{l-1}u\} \subset \tilde{\Omega}_E, \\ J_S^l(\tilde{\Omega}_E) &= \text{gr}_0(\tilde{\Omega}_E) \oplus \dots \oplus \text{gr}_l(\tilde{\Omega}_E) \subset \tilde{\Omega}_E. \end{aligned}$$

These spaces vanish for $l < 0$. The elements in gr_0 are called **constant**, those in gr_1 are called **linear**, while those in gr_l are called **homogeneous of degree l** . Similarly one defines $\text{gr}_l(\Omega_E)$; one has (e.g. using local formulas- see Remark 6)

$$(12) \quad \text{gr}_l(\Omega_E^{p,q}) = \Omega^p(S, \Lambda^q E \otimes S^l E^*),$$

where we regard the sections of E as fiberwise constant vertical vector fields on E and those of $S^l E^*$ as degree l homogeneous polynomial functions on E . Moreover, $\text{gr}_l(\tilde{\Omega}_E^{p,q})$ coincides with $\text{gr}_l(\Omega_E^{p,q})$ except for the case $l = 1, q = 1$ when

$$\text{gr}_1(\tilde{\Omega}_E^{p,1}) = \Omega^p(S, \mathfrak{X}_{\text{lin}}(E)),$$

where $\mathfrak{X}_{\text{lin}}(E)$ is the space of linear vector fields on E , i.e. projectable vector fields whose flow is fiberwise linear.

Our next aim is to introduce the partial derivative operators along S ,

$$d_S^l : \tilde{\Omega}_E \longrightarrow \text{gr}_l(\tilde{\Omega}_E),$$

used to introduce the jet operators j_S^n . To define and handle them, we use the formal power series expansion of $t\varphi_t(u)$ with respect to t . Although φ_t is not defined at $t = 0$, it is clear (use again Remark 6) that, for any $u \in \tilde{\Omega}_E$, the map

$$\mathbb{R}^* \ni t \mapsto t\varphi_t(u) \in \tilde{\Omega}_E$$

admits a smooth prolongation to \mathbb{R} . Hence the following definition makes sense.

Definition 3.8. For $u \in \tilde{\Omega}_E$ define the **n -th order derivatives** of u along S , denoted by $d_S^n u$, as the coefficients of the formal power expansion around $t = 0$:

$$\varphi_t(u) \cong t^{-1}u|_S + d_S u + t d_S^2 u + \dots$$

In other words,

$$d_S^n(u) = \frac{1}{n!} \frac{d^n}{dt^n} \big|_{t=0} t\varphi_t(u) \in \tilde{\Omega}_E.$$

For $n = 0$ we also use the notation $u|_S$. Define the **n -th order jet** of u along S as

$$j_S^n(u) = \sum_{k=0}^n d_S^k(u).$$

Lemma 3.9. We have that $d_S^n(u) \in \text{gr}_n(\tilde{\Omega}_E)$ and $j_S^n(u) \in J_S^n(\tilde{\Omega}_E)$.

Proof. Since $\varphi_r \circ \varphi_s = \varphi_{rs}$, we have that $\varphi_r(s\varphi_s(u)) = r^{-1}[\xi\varphi_\xi(u)]|_{\xi=rs}$. Taking the n -th derivative at $s = 0$, we obtain the first part, which implies the second. \square

The power series description, together with the properties of φ_t , are very useful in avoiding computations. For instance, using that φ_t preserves $[\cdot, \cdot]_\kappa$ we obtain:

Lemma 3.10. *For any $u, v \in \tilde{\Omega}_E$, $d_S^l[u, v]_{\ltimes} = \sum_{p+q=l+1} [d_S^p u, d_S^q v]_{\ltimes}$.*

As an illustration of our constructions let us look again at connections. We have already seen that an Ehresmann connection on E can be seen as an element $\Gamma \in \tilde{\Omega}_E^{1,1}$. Hence we can talk about its restriction to S as an element

$$\Gamma|_S \in \Omega^1(S, E).$$

This construction was introduced in [10] in a more ad-hoc fashion. Also, Γ is linear as an element of $\tilde{\Omega}_E$ if and only if it is a linear connection. For the direct implication: the properties of φ_t immediately imply that the \ltimes -bracket with Γ preserves $\text{gr}_0(\tilde{\Omega}_E^{\bullet,1}) = \Omega^\bullet(S, E)$ hence it induces a covariant derivative $d_\Gamma := [\Gamma, \cdot]_{\ltimes} : \Omega^\bullet(S, E) \longrightarrow \Omega^{\bullet+1}(S, E)$.

From now on we will restrict our attention to Poisson structures which admit S as a symplectic leaf. The following is immediate (see also Proposition 5.1 of [10]).

Lemma 3.11. *Let π be a horizontally non-degenerate Poisson structure on E with corresponding Dirac element $\gamma \in \tilde{\Omega}_E^2$ (cf. Proposition 3.5). Then S is a symplectic leaf if and only if $\gamma|_S$ lives in bi-degree $(2, 0)$. In this case the symplectic form is*

$$\omega_S := -\gamma|_S \in \text{gr}_0(\tilde{\Omega}_E^{2,0}) = \Omega^2(S).$$

For the first order approximation along S , we have:

Proposition 3.12. *Let π be a horizontally non-degenerate Poisson structure on E which admits S as symplectic leaf, with corresponding Dirac element $\gamma \in \tilde{\Omega}_E^2$. Then*

$$j_S^1 \gamma \in J_S^1(\tilde{\Omega}_E^2) \subset \tilde{\Omega}_E^2$$

is a Dirac element whose Poisson support N is an open neighborhood of S in E . In particular, on N , it is associated with a Poisson structure, denoted

$$j_S^1 \pi \in \mathfrak{X}^2(N).$$

Proof. The non-trivial part of the proposition (and which uses the fact that S is a symplectic leaf) is to show that $[j_S^1 \gamma, j_S^1 \gamma]_{\ltimes} = 0$. This follows by applying the similar equation for γ , using the Newton formula of Lemma 3.10 to compute its first order consequences and then using the fact that $\Omega(S)$ is in the center of Ω_E (Lemma 3.2) to delete the term $[\gamma|_S, d_S^2 \gamma]_{\ltimes}$. \square

Definition 3.13. *The Poisson bivector $j_S^1 \pi$ from the previous proposition is called the first order approximation of π along S .*

Remark 7. We explain how this definition is compatible with Proposition 1.10 (which says that the first order information of π along S is encoded in the Atiyah sequence of A_S) and with Definition 1.8 (the local model). Let us fix the symplectic structure ω_S of the leaf. Since the entire discussion depends only on $j_S^1 \pi$, we may assume that $\pi = j_S^1 \pi$, i.e. the corresponding Dirac element is the sum of

$$\pi^\vee \in \text{gr}_1(\tilde{\Omega}_E^{0,2}) = \Gamma(\Lambda^2 E \otimes E^*),$$

$$\Gamma = \Gamma_\pi \in \text{gr}_1(\tilde{\Omega}_E^{1,1}) = \Omega^1(S, \mathfrak{X}_{\text{lin}}(E)),$$

$$-\omega_S + \sigma = \mathbb{F}_\pi \in \text{gr}_0(\tilde{\Omega}_E^{2,0}) \oplus \text{gr}_1(\tilde{\Omega}_E^{2,0}) = \Omega^2(S) \oplus \Omega^2(S, E^*).$$

On the other hand, the algebroid structure is defined on $A_S = TS \oplus E^*$, with anchor the first projection. What is needed in order to describe such a Lie algebroid structure are precisely the elements π^\vee , Γ and σ . Explicitly, identifying

$$\Gamma(E^*) = \text{gr}_1(\Omega_E^{0,0}),$$

one obtains the following formulas for the bracket $[\cdot, \cdot]_A$ on A_S :

$$[\alpha, \beta]_A = [\beta, [\pi^\vee, \alpha]_\kappa]_\kappa, \quad [\alpha, X]_A = [\Gamma, \alpha]_\kappa(X), \quad [X, Y]_A = [X, Y] + \sigma(X, Y),$$

where $\alpha, \beta \in \Gamma(E^*)$, $X, Y \in \mathfrak{X}(S)$. This describes the 1-1 correspondence between first jets of π 's, with (S, ω_S) as a symplectic leaf, and Lie algebroid structures on A_S with anchor the first projection- as the explicit version of our Proposition 1.10 and a more compact description of Theorem 4.1 in [27].

To explain the compatibility with Definition 1.8, we first use Remark 3 which described the local model as a Dirac structure on the dual K^* of the kernel of A_S , using a splitting $\theta : A_S \rightarrow K$. In our case, $K = E^*$ (hence the Dirac structure is on E), one can use as splitting the canonical projection and, since all the objects involved are given by explicit formulas, it is straightforward to compute and compare the resulting Dirac structure (coming from Remark 3) and the Dirac structure corresponding to $j_S^1\pi$: the two coincide!

The following proposition is important for the Moser path method. We connect π to its first order approximation $j_S^1\pi$ by a smooth path of Poisson structures.

Proposition 3.14. *Let π be a horizontally non-degenerate Poisson structure on E which admits S as a symplectic leaf. Let $\gamma \in \tilde{\Omega}_E^2$ be the corresponding Dirac element. Then, on a small enough neighborhood N of S , there exists a smooth path of Poisson structures*

$$\pi_t \in \mathfrak{X}^2(N), \quad \text{with } \pi_1 = \pi|_N, \quad \pi_0 = j_S^1(\pi).$$

More precisely, π_t can be chosen via the smooth path of Dirac elements

$$\gamma_t = \gamma|_S + \frac{t\varphi_t(\gamma) - \gamma|_S}{t} \quad \forall t \in (0, 1],$$

and N is the intersection of the Poisson supports of these elements.

Proof. Note first that γ_t extends smoothly at $t = 0$ as $\gamma_0 = j_S^1\gamma$. We will perform the computations for $t \neq 0$ and extended to $t = 0$ by continuity. We write:

$$\gamma_t = \varphi_t(\gamma) + (t^{-1} - 1)\omega_S = \varphi_t(\gamma + (1 - t)\omega_S).$$

Using that φ_t commutes with the bracket, γ is a Dirac element and $[\omega_S, \omega_S]_\kappa = 0$:

$$[\gamma_t, \gamma_t]_\kappa = \varphi_t([\gamma + (1 - t)\omega_S, \gamma + (1 - t)\omega_S]_\kappa) = 2(1 - t)\varphi_t([\gamma, \omega_S]_\kappa).$$

By Lemma 3.2, formula (9), and the fact that ω_S is symplectic we have that

$$[\gamma, \omega_S]_\kappa = \mathcal{L}_{p_S(\gamma)}(\omega_S) = \mathcal{L}_{\gamma_S}(\omega_S) = d\omega_S = 0.$$

This proves that $[\gamma_t, \gamma_t]_\kappa = 0$. It is also clear that $p_S(\gamma_t) = p_S(\varphi_t(\Gamma_\pi)) = p_S(\Gamma_\pi) = \gamma_S$. Finally, note that $\gamma_t|_S = -\omega_S$, hence (S, ω_S) is a symplectic leaf for all γ_t 's. This also show that $S \subset N$. To see that N is indeed open, we apply the topological tube lemma: from the smoothness of the family γ_t , \mathcal{N} consisting of pairs (t, e) with e in the support of γ_t is open in $[0, 1] \times E$; also, $e \in N$ means $[0, 1] \times \{e\} \subset \mathcal{N}$ hence, by the tube lemma, $[0, 1] \times W \subset \mathcal{N}$ (hence $W \subset N$) for some open $W \ni e$. \square

4. PROOF OF THE MAIN THEOREM; STEP 1: MOSER PATH METHOD

In this section we use the Moser path method to reduce the proof of the main theorem to some cohomological equations. The main outcome is Theorem 4.1 below.

Let (M, π) is a Poisson manifold and let (S, ω_S) be a symplectic leaf. We start by describing the relevant cohomologies. They are all relatives of the Poisson cohomology groups $H_\pi^\bullet(M)$ - defined by the complex $(\mathfrak{X}^\bullet(M), d_\pi)$ where $d_\pi = [\pi, \cdot]$. The first one is, intuitively, the Poisson cohomology of the germ of (M, π) around S :

$$H_\pi^\bullet(M)_S = \lim_{S \subset U} H_{\pi|_U}^\bullet(U),$$

where the limit is the direct limit over all the open neighborhoods U of S in M . The next relevant cohomology, the **Poisson cohomology restricted to S** , denoted

$$H_{\pi,S}^{\bullet}(M),$$

is defined by the complex $(\mathfrak{X}_{|S}^{\bullet}(M), d_{\pi|S})$, where $\mathfrak{X}_{|S}(M) = \Gamma(\Lambda^{\bullet}TM|_S)$.

The last relevant cohomology is a version of $H_{\pi,S}^{\bullet}(M)$ with coefficients in the conormal bundle of S , which is best described using Lie algebroids. Given a Lie algebroid A and a representation (V, ∇) of A , the cohomology of A with coefficients in V , denoted $H^{\bullet}(A, V)$, is defined by the complex $\Omega^{\bullet}(A, V) = \Gamma(\Lambda^{\bullet}A^* \otimes V)$ endowed with the differential $d_{A, \nabla}$ given by the classical Koszul formula (see [21]). When V is the trivial 1-dimensional representation, one obtains the cohomology DeRham cohomology of A . For instance, $H_{\pi}^{\bullet}(M)$ is just the cohomology of the cotangent algebroid T^*M , while $H_{\pi,S}^{\bullet}(M)$ is just the cohomology of the restriction

$$A_S := T^*M|_S.$$

The conormal bundle ν_S^* of S is a representation of A_S with $\nabla_{\alpha}(\beta) = [\alpha, \beta]$ (where $[\cdot, \cdot]$ is the bracket of A_S). The last relevant cohomology is

$$H_{\pi,S}^{\bullet}(M, \nu_S^*) := H^{\bullet}(A_S, \nu_S^*).$$

Theorem 4.1. *Let S be an embedded symplectic leaf of a Poisson manifold (M, π) and let $j_S^1\pi$ be the first order approximation of π along S associated to some tubular neighborhood of S in M . If*

$$H_{\pi}^2(M)_S = 0, \quad H_{\pi,S}^1(M) = 0, \quad H_{\pi,S}^1(M, \nu_S^*) = 0,$$

then, around S , π and $j_S^1\pi$ are Poisson diffeomorphic, by a Poisson diffeomorphism which is the identity on S .

The rest of this section is devoted to the proof of the theorem, followed by a slight improvement that will be used in order to prove Proposition 2.1.

First of all, by using a tubular neighborhood, we may assume that $M = E$ is a vector bundle E over S and π is horizontally non-degenerate. Let $\gamma \in \tilde{\Omega}_E^2$ be the associated Dirac element. We first rewrite in terms of E and γ the complexes computing the last two cohomologies in the statement of the theorem.

Lemma 4.2. *For any $l \geq 0$, the complex $(\Omega^{\bullet}(A_S, S^l E^*), d_A)$ computing the cohomology of A_S with coefficients in the l -th symmetric power of $\nu_S^* = E^*$ is canonically isomorphic to the complex $(gr_l(\Omega_E^{\bullet}), [d_S^1 \gamma, \cdot])$.*

Proof. We use the notations and the explicit formulas from Remark 7. With the identification $A_S = TS \oplus E^*$ and the identification (12), we see that

$$gr_l(\Omega_E^{\bullet}) = \Omega^{\bullet}(A_S, S^l E^*).$$

Hence we still have to show that the two differentials coincide. Let d_l be the differential for the cohomology of A_S with coefficients in $S^l E^*$. Denote

$$\delta := d_S^1 \gamma = \pi^{\vee} + \Gamma + \sigma.$$

Since both d_l and ad_{δ} act as derivations and d_l is the l -th symmetric power of d_1 , it is enough to prove that for $\omega \in \Omega(S)$, $V \in \Gamma(E)$ and $\psi \in \Gamma(E^*)$, we have that

$$(13) \quad d_0(\omega) = [\delta, \omega]_{\times}, \quad d_0(V) = [\delta, V]_{\times}, \quad d_1(\psi) = [\delta, \psi]_{\times}.$$

For the first equation, observe that by Lemma 3.2 and (9), we have that $[\delta, \omega]_{\times} = \mathcal{L}_{\gamma_S} \omega = d\omega$ and since the anchor is a Lie algebroid map, we also have that $d_0 \omega = d\omega$.

In the computations below, we use that, for $W \in \Gamma(\Lambda^{\bullet} E)$ and $\eta \in \Gamma(E^*)$, we have that $W(\eta, \cdot) = -[\eta, W]$, where on the left hand side we just contract W with η and on the right hand side we regard W and η as elements in $\mathfrak{X}^{\bullet}(E)$. Let $V \in \Gamma(E)$.

For $\eta, \psi \in \Gamma(E^*)$, using that $C^\infty(S)$ commutes with Ω_E and that the only term in $[\delta, V]_\times$ which is nonzero on $\Gamma(E^*) \times \Gamma(E^*)$ is $[\pi^\vee, V]$, we have:

$$\begin{aligned} d_0(V)(\eta, \psi) &= -V([\eta, \psi]_A) = [[\psi, [\pi^\vee, \eta]], V] = \\ &= [\psi, [[\pi^\vee, V], \eta]] + [\psi, [\pi^\vee, [\eta, V]]] + [[\psi, V], [\pi^\vee, \eta]] = \\ &= [\pi^\vee, V](\eta, \psi) - [\psi, [\pi^\vee, V(\eta)]] - [V(\psi), [\pi^\vee, \eta]] = \\ &= [\pi^\vee, V](\eta, \psi) = [\delta, V]_\times(\eta, \psi). \end{aligned}$$

For $X \in \mathfrak{X}(S)$ and $\eta \in \Gamma(E^*)$, we have that

$$\begin{aligned} d_0(V)(X, \eta) &= [X, V(\eta)] - V([X, \eta]_A) = [X, [V, \eta]] - [V, [\text{hor}_\Gamma(X), \eta]] = \\ &= [\text{hor}_\Gamma(X), [V, \eta]] - [V, [\text{hor}_\Gamma(X), \eta]] = [\text{hor}_\Gamma(X), V](\eta) = \\ &= [\Gamma, V]_\times(X, \eta) = [\delta, V]_\times(X, \eta), \end{aligned}$$

where we have used the fact that the only term in $[\delta, V]_\times$ which is nonzero on $\mathfrak{X}(S) \times \Gamma(E^*)$ is $[\Gamma, V]_\times$. Consider now $X, Y \in \mathfrak{X}(S)$. We have that

$$\begin{aligned} d_0(V)(X, Y) &= -V([X, Y]_A) = -V(\sigma(X, Y)) = -[\sigma(X, Y), V] = \\ &= [V, \sigma](X, Y) = [\delta, V]_\times(X, Y), \end{aligned}$$

where we have used the fact that the only term in $[\delta, V]_\times$ which is nonzero on $\mathfrak{X}(S) \times \mathfrak{X}(S)$ is $[\sigma, V]_\times$. Hence we have proven the second equation in (13).

Let $\psi \in \Gamma(E^*)$. For $\eta \in \Gamma(E^*)$ and respectively $X \in \mathfrak{X}(S)$, we have that

$$\begin{aligned} d_1(\psi)(\eta) &= -[\psi, \eta]_A = -[\eta, [\pi^\vee, \psi]] = [\pi^\vee, \psi](\eta) = [\delta, \psi]_\times(\eta), \\ d_1(\psi)(X) &= [X, \psi]_A = [\text{hor}_\Gamma(X), \psi] = [\Gamma, \psi]_\times(X) = [\delta, \psi]_\times(X), \end{aligned}$$

and this proves also the last equation in (13). \square

We now return to the proof of Theorem 4.1. We will use the path of Poisson structures π_t provided by Proposition 3.14 and the associated Dirac elements γ_t , with $\gamma_1 = \gamma$ corresponding to π and $\gamma_0 = j_S^1 \gamma$. The first part of the proof holds for general paths γ_t . We are looking for a family μ_t of diffeomorphisms defined on a neighborhood of S in E , for all $t \in [0, 1]$, such that $\mu_t = \text{Id}_S$ on S , $\mu_0 = \text{Id}$ and

$$(14) \quad \mu_t^* \pi_t = j_S^1 \pi$$

for all $t \in [0, 1]$. Then μ_1 will be the desired isomorphism. We will define μ_t as the flow of a time depending vector field Z_t , i.e. as the solution of:

$$\frac{d}{dt} \mu_t(x) = Z_t(\mu_t(x)), \quad \mu_0(x) = x.$$

Hence we are looking for the time dependent Z_t defined on a neighborhood of S in E . The first condition we require is that $Z = 0$ along S . This implies that $\mu_t = \text{Id}$ on S and that μ_t 's are well-defined up to time 1 on an open neighborhood U of S , i.e. $[0, 1] \times U$ is inside the domain \mathcal{D} of the flow of Z_t . For the last assertion one uses again the tube lemma: for each $x \in S$, since $[0, 1] \times \{x\} \subset \mathcal{D}$, one can find an open $U_x \ni x$ with $[0, 1] \times U_x \subset \mathcal{D}$ and one takes $U = \cup_x U_x$.

Finally, since (14) holds at $t = 0$, it suffices to require its infinitesimal version:

$$(15) \quad \mathcal{L}_{Z_t}(\pi_t) + \dot{\pi}_t = 0.$$

The next step is to rewrite this equation in terms of the Dirac elements γ_t . For each t we consider the isomorphism induced by γ_t (see Proposition 3.6):

$$\tau_t : (\mathfrak{X}^\bullet(E), d_{\pi_t}) \longrightarrow (\Omega_E^\bullet, ad_{\gamma_t}).$$

Strictly speaking, the map τ_t is defined only on the open N from Proposition 3.14, but since the discussion is local around S , we allow ourselves this notational sloppiness. Finding the Z_t 's is equivalent to finding

$$V_t := \tau_t(Z_t) \in \Omega_E^1.$$

The following is our compact version of Proposition 2.14 of [28].

Lemma 4.3. *The homotopy equation (15) is equivalent to the following equation:*

$$(16) \quad [\gamma_t, V_t]_{\times} = \dot{\gamma}_t \quad (\forall) \quad t \in [0, 1]$$

required to hold on a neighborhood of S in E .

Proof. If we decompose $Z_t = X_t + Y_t$, into its Γ_{π_t} -horizontal and vertical components, then, using the explicit form of τ_t from (11), it follows that V_t is the sum of two elements of bi-degree $(1, 0)$ and $(0, 1)$ respectively:

$$V_t = Y_t - \mathbb{F}_{\pi_t}^{\sharp}(X_t).$$

Therefore (16) breaks down into the following list of equations, of various degrees:

$$[\pi_t^{\vee}, Y_t]_{\times} = \dot{\pi}_t^{\vee}, \quad [\Gamma_{\pi_t}, Y_t]_{\times} - [\pi_t^{\vee}, \mathbb{F}_{\pi_t}^{\sharp}(X_t)]_{\times} = \dot{\Gamma}_{\pi_t}, \quad [\mathbb{F}_{\pi_t}, Y_t]_{\times} - [\Gamma_{\gamma_t}, \mathbb{F}_{\pi_t}^{\sharp}(X_t)]_{\times} = \dot{\mathbb{F}}_{\pi_t}.$$

These are precisely the equations appearing in Proposition 2.14 of [28]. \square

In conclusion, we are looking for elements $V_t \in \Omega_E^1$, defined for all $t \in [0, 1]$ on some open neighborhood of S in E , with the property that $V_t|_S = 0$ and satisfying the equations (16). There is one equation for each t but, since γ_t is of a special type, one can reduce everything to a single equation.

Lemma 4.4. *Assume that there exists $X \in \Omega_E^1$ such that $j_S^1 X = 0$ and*

$$(17) \quad [\gamma, X]_{\times} = \dot{\gamma}_1.$$

Then $V_t := t^{-1}\varphi_t(X)$ satisfies the homotopy equations (16).

Proof. The condition that the first jet of X along S vanishes ensures that V_t is a smooth family defined also at $t = 0$ and that V_t vanishes along S . We check the homotopy equations at all $t \in (0, 1]$. For the left hand side:

$$[\gamma_t, V_t]_{\times} = [\varphi_t(\gamma) + (1 - t^{-1})\omega_S, V_t]_{\times} = \varphi_t([\gamma, \varphi_{t^{-1}}(V_t)]_{\times}) = t^{-1}\varphi_t([\gamma, X]_{\times}),$$

where we have used the fact that ω_S lies in the center of Ω_E and that φ_t commutes with the brackets. Using the assumption on X , we find

$$[\gamma_t, V_t]_{\times} = t^{-1}\varphi_t(\dot{\gamma}_1).$$

Hence (16) will follow for $t \in (0, 1]$ if we prove the following equation

$$(18) \quad t^{-1}\varphi_t(\dot{\gamma}_1) = \dot{\gamma}_t.$$

Note first that the explicit formula for γ_t implies that

$$\varphi_t(\gamma_s) = \gamma_{ts} + (1 - t^{-1})\omega_S.$$

Taking the derivative of this equation at $s = 1$, we obtain the result. \square

The equation for X in the last lemma lives in the cohomology of $(\Omega_E^{\bullet}, \text{ad}_{\gamma})$. Note that the right hand side of the equation is indeed closed. This follows by taking the derivative with respect to t at $t = 1$ in $[\gamma_t, \gamma_t]_{\times} = 0$. Using Proposition 3.6 and the first assumption of Theorem 4.1 we see that, after eventually shrinking its domain of definition, $\dot{\gamma}_1$ exact. This ensures the existence of X . To conclude the proof of the theorem, we still have to show that X can be corrected so that $j_S^1 X = 0$. We will do so by finding $F \in \Omega_E^0 = C^{\infty}(E)$, such that

$$j_S^1([\gamma, F]_{\times}) = j_S^1(X).$$

Then $X' = X - [\gamma, F]_\times$ will be the correction of X . Since the condition only depends on $j_S^1 F$, it suffices to look for F of type

$$F = F_0 + F_1 \in \text{gr}_0(\Omega_E^0) \oplus \text{gr}_1(\Omega_E^0) = C^\infty(S) \oplus \Gamma(E^*).$$

So $F|_S = F_0$, $d_S^1 F = F_1$. To compute $j_S^1([\gamma, F]_\times)$, we write:

$$\begin{aligned} \varphi_t([\gamma, F]_\times) &= [\varphi_t(\gamma), \varphi_t(F)]_\times = [-t^{-1}\omega_S + d_S^1\gamma + td_S^2\gamma + t^2(\dots), t^{-1}F_0 + F_1]_\times = \\ &= t^{-1}[d_S^1\gamma, F_0]_\times + [d_S^1\gamma, F_1]_\times + t(\dots), \end{aligned}$$

where we have used that ω_S and F_0 commute with Ω_E (Lemma 3.2). So we have to solve the following cohomological equations:

$$(19) \quad [d_S^1\gamma, F_0]_\times = X|_S, \quad [d_S^1\gamma, F_1]_\times = d_S^1 X.$$

By Lemma 4.2, the relevant cohomologies are precisely the ones assumed to vanish in the theorem. So it is enough to show that $X|_S$ and $d_S^1 X$ are closed with respect to the differential $[d_S^1\gamma, \cdot]_\times$. These are precisely the first order consequences of the equation (17) that X satisfies. By (18), we have that $\varphi_t(\dot{\gamma}_1) = t\dot{\gamma}_t$, hence $j_S^1(\dot{\gamma}_1) = 0$. So the Newton formula of Lemma 3.10 applied to (17) gives:

$$(20) \quad [d_S^1\gamma, X|_S]_\times = 0, \quad [d_S^1\gamma, d_S^1 X]_\times + [d_S^2\gamma, X|_S]_\times = 0.$$

Hence $X|_S$ is closed, and so we can find F_0 satisfying the first equation in (19). In particular, using also Lemma 3.2 this shows that

$$X|_S = [d_S^1\gamma, F_0]_\times = dF_0 \in \Omega(S).$$

Hence $X|_S$ commutes with Ω_E , and since $d_S^2\gamma \in \Omega_E$, the second equation of (20) becomes $[d_S^1\gamma, d_S^1 X]_\times = 0$. This finishes the proof.

Remark 8. The previous arguments reveal a certain cohomology class related to the linearization problem, which we will describe now more explicitly. First of all, π itself defines a class $[\pi] \in H_\pi^2(M)_S$. Consider a tubular neighborhood $p: E \rightarrow S$. Since any two tubular neighborhoods are isotopic, it follows that the class

$$[p^*(\omega_S)] \in H_{\text{dR}}^\bullet(M)_S = \lim_{S \subset U} H_{\text{dR}}^\bullet(U),$$

is independent of p . On the other hand, π gives a chain map between the complexes

$$\wedge^\bullet \pi|_U^\# : (\Omega^\bullet(U), d) \rightarrow (\mathfrak{X}^\bullet(U), d_\pi).$$

Let $[\pi|_S] \in H_\pi^2(M)_S$ be the image of $[p^*(\omega_S)]$ under the induced map in cohomology.

Definition 4.5. The linearization class associated to an embedded leaf S of a Poisson manifold (M, π) is defined by

$$l_{\pi, S} := [\pi] - [\pi|_S] \in H_\pi^2(M)_S.$$

It would be interesting to study this class in more detail.

Corollary 4.6. In the previous theorem, the condition $H_\pi^2(M)_S = 0$ can be replaced by the condition $l_{\pi, S} = 0$.

Proof. In the proof above we only used the vanishing of $[\dot{\gamma}_1]$, hence, by Proposition 3.6, it suffices to show that $[\tau_\pi^{-1}(\dot{\gamma}_1)] = l_{\pi, S}$. By the definition of φ_t , we have that

$$\dot{\gamma}_t = (t^{-1} - 1)\omega_S + t^{-1}m_t^*(\mathbb{F}_\pi) + m_t^*(\Gamma_\pi) + tm_t^*(\pi^\vee).$$

Since m_{e^t} is the flow of the Liouville vector field $\mathcal{E} = \sum y_i \frac{\partial}{\partial y_i}$, we obtain

$$\dot{\gamma}_1 = -\omega_S - \mathbb{F}_\pi + \pi^\vee + [\mathcal{E}, \gamma_1]_\times.$$

Using (10), (11) and the fact that π^\vee is vertical, we obtain the conclusion

$$\tau_\pi^{-1}(\pi^\vee - \mathbb{F}_\pi - \omega_S) = \pi^\vee + \pi^h - \wedge^2 \pi^\#(\omega_S) = \pi - \wedge^2 \pi^\#(\omega_S).$$

□

5. PROOF OF THE MAIN THEOREM; STEP 2: INTEGRABILITY

In this section we show that the conditions of our main theorem imply the integrability of the Poisson structure around the symplectic leaf which, in turn, implies that the cohomological conditions from Theorem 4.1 are satisfied. As in [11], this step will be divided into three sub-steps (the three subsections of this section):

- Step 2.1: Integrability implies the needed cohomological conditions.
- Step 2.2: Existence of a “nice” symplectic realization implies integrability.
- Step 2.3: Prove the existence of such “nice” symplectic realizations.

In the first sub-step we will also finish the proof of Proposition 2.1.

5.1. Step 2.1: Reduction to integrability. As promised:

Theorem 5.1. *Let (M, π) be a Poisson manifold, $x \in M$, and S be the symplectic leaf through x . If P_x , the homotopy bundle at x , is smooth and compact, then*

$$H_{\pi, S}^1(M) = 0, \quad H_{\pi, S}^1(M, \nu_S^*) = 0.$$

If moreover $H^2(P_x) = 0$ and S admits an open neighborhood U whose associated groupoid $\Sigma(U, \pi|_U)$ is smooth and Hausdorff, then also

$$H_\pi^2(M)_S = 0.$$

Proof. The first part of the proof is completely similar to that of Theorem 2 in [11]: a consequence of the Van Est isomorphism and the vanishing of differentiable cohomology for proper groupoids. More precisely: the conditions on P_x imply that the groupoid $\mathcal{G}(A_S)$ of $A_S = T^*M|_S$ is smooth and compact (hence the differentiable cohomology with coefficients vanishes); since its s -fibers are 1-connected, the Van Est map with coefficients is an isomorphism in degrees 1 and 2; hence the cohomology of $T^*M|_S$ in degrees 1 and 2 with any coefficients vanishes.

For the second part, let $\Sigma(U) = \Sigma(U, \pi|_U)$ be the symplectic groupoid integrating $(U, \pi|_U)$. It suffices to show that, for any open $W \subset U$ containing S , there exists a smaller one V such that $H_\pi^2(V) = 0$. Proceeding as in the first part, it suffices to produce V 's for which $\Sigma(V) = \Sigma(V, \pi|_V)$ has s -fibers which are compact and cohomologically 2-connected. Let $\mathcal{G} \subset \Sigma(U)$ be the set of arrows with source and target inside W and for which both the s -fiber and the t -fiber are diffeomorphic to P_x . By local Reeb stability applied to the foliation by the s -fibers (and t -fibers respectively), we see that all four conditions are open, therefore $\mathcal{G} \subset \Sigma(U)$ is open, and by assumption, all arrows above S are in \mathcal{G} . By the way \mathcal{G} was defined, we see that it is an open subgroupoid over the invariant open $V := s(\mathcal{G})$. So $\mathcal{G} = \Sigma(V)$ and it has all the desired properties. \square

End of the proof of Proposition 2.1. We will adapt the previous proof, making use of Remark 8. We have to show that

$$[\pi] - [\pi|_S] = 0 \in H_\pi^2(M)_S.$$

We show that, for any tubular neighborhood $p : W \rightarrow S$ of S with $W \subset U$, there exists a smaller one V such that

$$[\pi] - [\pi|_S] = 0 \in H_\pi^2(V).$$

Let V be as in the previous proof. We may assume that V is connected; if not, we replace it by the component containing S . Since $\Sigma(V)$ is still proper, it suffices to show that the class above is in the image of the Van Est map of $\Sigma(V)$ (in degree 2). By Corollary 2 in [6], this image consists of elements $[\omega] \in H_\pi^2(V)$ for which $\int_\gamma \omega = 0$, for all 2-spheres γ in the s -fibers of $\Sigma(V)$. Hence it suffices to show that $[\pi] - [\pi|_S]$ satisfies this condition. As before, let $\tilde{\omega}_S = p^*\omega_S$; also consider

the symplectic form Ω on the symplectic groupoid. The right invariant 2-form on the s -fibers of $\Sigma(V)$ corresponding to $\tilde{\omega}_S$ is $t^*(\tilde{\omega}_{S|V})$ restricted to the s -fibers. The right invariant 2-form on the s -fibers of $\Sigma(V)$ corresponding to the class $[\pi]$ is the pullback by t to the s -fibers of the symplectic structure on the leaves of $(V, \pi|_V)$; on the other hand, it is also the restriction to the s -fibers of Ω . In particular the two coincide on the s -fibers above S . For the correspondence between Poisson cocycles and right invariant, foliated 2-forms the symplectic groupoid, see [31]. Consider

$$\omega := \Omega - t^*(\tilde{\omega}_{S|V}).$$

Let γ a 2-sphere in an s -fiber of $\Sigma(V)$. Since V is connected, we can find a homotopy between γ and a 2-sphere γ_1 which lies in an s -fiber over S . Since ω is closed, we have that $\int_\gamma \omega = \int_{\gamma_1} \omega$, and since the restriction of ω to s -fibers over S vanishes, it follows that $\int_{\gamma_1} \omega = 0$. This implies the conclusion. \square

5.2. Step 2.2: Reduction to the existence of "nice" symplectic realizations. Next, we show that the integrability condition required in the last theorem is implied by the existence of a symplectic realization with some specific properties.

We will use the following notation. Given a symplectic realization μ , we denote by $\mathcal{F}(\mu)$ the foliation defined by μ , identified also with the involutive distribution $\text{Ker}(d\mu)$. Its symplectic orthogonal is a new distribution $\mathcal{F}(\mu)^\perp$. Since μ is a Poisson map, it is well-known (and follows easily) that $\mathcal{F}(\mu)^\perp$ is also involutive.

Theorem 5.2. *Let (M, π) be a Poisson manifold and let S be a symplectic leaf. Assume that there exists a symplectic realization*

$$\mu : (\Sigma, \Omega) \longrightarrow (U, \pi|_U)$$

of some open neighborhood U of S in M such that any leaf of the foliation $\mathcal{F}^\perp(\mu)$ which intersects $\mu^{-1}(S)$ is compact and 1-connected.

Then there exists an open neighborhood $V \subset U$ of S such that the Weinstein groupoid $\Sigma(V, \pi|_V)$ is Hausdorff and smooth.

Proof. We may assume that all leaves of $\mathcal{F}(\mu)^\perp$ are compact and 1-connected. Otherwise, we replace Σ by Σ' and U by $U' = \mu(\Sigma')$, where Σ' is defined as the set of points $y \in \Sigma$ with the property that the leaf of $\mathcal{F}(\mu)^\perp$ through y is compact and 1-connected. Local Reeb stability implies that Σ' is open in Σ . The hypothesis implies that $\mu^{-1}(S) \subset \Sigma'$ and, since μ is open, U' is an open neighborhood of S .

Clearly, we may also assume that $U = M$. Hence we have a symplectic realization

$$\mu : (\Sigma, \Omega) \longrightarrow (M, \pi)$$

with the property that all the leaves of $\mathcal{F}(\mu)^\perp$ are compact and 1-connected. We claim that $\Sigma(M, \pi)$ has the desired properties. By Theorem 8 in [8], if the symplectic realization μ is complete, then $\Sigma(M, \pi)$ is smooth. The compactness assumption on the leaves of $\mathcal{F}(\mu)^\perp$ implies that μ is complete since the Hamiltonian vector fields of type $X_{\mu^*(f)}$ are tangent to these leaves. For Hausdorffness, we take a closer look to the argument of [8]. It is based on a natural isomorphism of groupoids

$$\Sigma(M, \pi) \times_M \Sigma \cong \mathcal{G}(\mathcal{F}(\mu)^\perp),$$

where the left hand side is the fibered product over s and μ , and the right hand side is the homotopy groupoid of the foliation $\mathcal{F}(\mu)^\perp$. Since homotopy groupoids are always smooth, [8] concluded that $\Sigma(M, \pi)$ is smooth. In our case, the leaves of $\mathcal{F}(\mu)^\perp$ are 1-connected, hence the homotopy groupoid is a subgroupoid of $M \times M$. So it is Hausdorff, from which it follows easily that also $\Sigma(M, \pi)$ is Hausdorff. \square

5.3. Constructing symplectic realizations from transversals in the manifold of cotangent paths. In this subsection we describe a general method for constructing symplectic realizations; it will be used in the next subsection to produce a symplectic realization with the properties required to apply Theorem 5.2.

In this and the next subsection we will use the same notations as in the proofs of the main results of [7, 8, 11] and also some familiarity with those might be useful.

Throughout this subsection, (M, π) is an arbitrary Poisson manifold. We know that if (M, π) is integrable, then the source map of $\Sigma(M, \pi)$ produces a complete symplectic realization. It may be helpful to have in mind that, although (M, π) may fail to be integrable, i.e. $\Sigma(M, \pi)$ may fail to be smooth, there is always a “local groupoid” $\Sigma_{\text{loc}}(M, \pi)$ which is smooth and produces a symplectic realization of (M, π) (but fails to be complete). The plan is to analyze closer the explicit construction of $\Sigma(M, \pi)$ and of its symplectic form to produce other symplectic realizations, sitting in between $\Sigma(M, \pi)$ and $\Sigma_{\text{loc}}(M, \pi)$. They will have a better chance of both being smooth and having the desired properties.

We will use the same notations as in [7, 8]. We consider:

- $\tilde{\mathcal{X}} = \tilde{P}(T^*M)$ is the space of all C^2 -paths in T^*M . Recall [7] that $\tilde{\mathcal{X}}$ has a natural structure of Banach manifold.
 - $\mathcal{X} = P(T^*M)$ is the space of all cotangent paths which, by Lemma 4.6 in [7], is a Banach submanifold of $\tilde{\mathcal{X}}$.
 - $\mathcal{F} = \mathcal{F}(T^*M)$ is the foliation on \mathcal{X} given by the equivalence relation of cotangent homotopy; it is a smooth foliation on \mathcal{X} of finite codimension.
- In (24) we will recall the description of \mathcal{F} via involutive distributions.

Thinking of $\tilde{\mathcal{X}}$ as the cotangent space of the space $P(M)$ of paths in M , it comes with a canonical symplectic structure $\tilde{\Omega}$. To avoid issues regarding symplectic structures on Banach manifolds let us just define $\tilde{\Omega}$ explicitly:

$$\tilde{\Omega}(X_a, Y_a) = \int_0^1 \omega_{\text{can}}(X_a, Y_a)_{a(t)} dt, \text{ for } a \in \tilde{\mathcal{X}}, X, Y \in T_a \tilde{\mathcal{X}},$$

where X_a, Y_a are interpreted as paths in $T(T^*M)$ sitting above a and where ω_{can} is the canonical symplectic form on T^*M . It can be checked directly that $\tilde{\Omega}$ is closed; we only need its restriction to \mathcal{X} :

$$\Omega := \tilde{\Omega}|_{\mathcal{X}} \in \Omega^2(\mathcal{X}).$$

We will prove that the kernel of Ω is precisely \mathcal{F} and Ω is invariant under the holonomy of \mathcal{F} ; this ensures that Ω descends to a symplectic form on the leaf space $\Sigma(M, \pi)$ (whenever smooth). Our strategy is a variation of this idea: we look at transversals T of the foliation and equivalence relations \sim on T which are weaker than the holonomy; by the same arguments, Ω descends to a symplectic form on T/\sim , provided this quotient is smooth. Our job will be to produce \sim .

We fix a torsion-free connection ∇ on M . Then a tangent vector X to T^*M can be interpreted as a pair (\overline{X}, θ_X) , where $\overline{X} = (dp)(X) \in TM$ and $\theta_X \in T^*M$ is the vertical component with respect to ∇ . For torsion-free connections, the horizontal distribution on T^*M is Lagrangian with respect to ω_{can} ; it follows that:

$$\omega_{\text{can}}(X, Y) = \langle \theta_Y, \overline{X} \rangle - \langle \theta_X, \overline{Y} \rangle,$$

Similarly, a tangent vector $X \in T_a \tilde{\mathcal{X}}$ is represented by a pair (\overline{X}, θ_X) , where \overline{X} is a C^1 -path in TM , θ_X in T^*M , both sitting above the base path $\gamma = p \circ a$; also,

$$\tilde{\Omega}(X, Y) = \int_0^1 (\langle \theta_Y, \overline{X} \rangle - \langle \theta_X, \overline{Y} \rangle) dt.$$

To describe $T\mathcal{X}$ using ∇ , one uses two T^*M -connections: one on T^*M and one on TM , both denoted $\bar{\nabla}$: for $\alpha, \beta \in \Omega(M)$ and $X \in \mathfrak{X}(M)$,

$$\bar{\nabla}_\alpha(\beta) = \nabla_{\pi^\sharp(\beta)}(\alpha) + [\alpha, \beta]_\pi, \quad \bar{\nabla}_\alpha(X) = \pi^\sharp(\nabla_X(\alpha)) + [\pi^\sharp(\alpha), X].$$

Recall that $[\cdot, \cdot]_\pi$ is given by (2). Note that the two connections are related by

$$(21) \quad \bar{\nabla}_\alpha(\pi^\sharp(\beta)) = \pi^\sharp(\bar{\nabla}_\alpha(\beta)).$$

Since ∇ is torsion-free, it follows that they also satisfy the duality relation:

$$(22) \quad \langle \bar{\nabla}_\alpha(\beta), X \rangle + \langle \beta, \bar{\nabla}_\alpha(X) \rangle = \pi^\sharp(\alpha)(\langle \beta, X \rangle).$$

Given a cotangent path a with base path γ and a C^2 -path U in T^*M or TM above γ , one has the induced derivative $\bar{\nabla}_a(U)$ of U along a - a C^1 -path above γ , sitting in the same space as U (T^*M or TM). Explicitly, choosing a time depending section \tilde{U} of class C^2 such that $\tilde{U}_t(\gamma(t)) = U(t)$,

$$\bar{\nabla}_a(U)(x) = \nabla_a \tilde{U}_t(x) + \frac{d\tilde{U}_t}{dt}(x), \text{ at } x = \gamma(t).$$

With these, the tangent space

$$T_a\mathcal{X} \subset T_a\tilde{\mathcal{X}}$$

corresponds to those pairs (\bar{X}, θ_X) satisfying (see [7]):

$$\bar{\nabla}_a(\bar{X}) = \pi^\sharp(\theta_X).$$

Note that, the condition that \bar{X} and θ_X are of class C^1 , together with the equation above, forces \bar{X} to be of class C^2 . Using equation (22), it is straightforward to show that for $a \in \mathcal{X}$, the two derivatives $\bar{\nabla}_a$ on TM and T^*M are related by:

$$(23) \quad \langle \bar{\nabla}_a(\theta), V \rangle + \langle \theta, \bar{\nabla}_a(V) \rangle = \frac{d}{dt} \langle \theta, V \rangle,$$

for all paths θ in T^*M and V in TM , both sitting over $\gamma = p \circ a$.

To finally define the distribution $\mathcal{F} \subset T\mathcal{X}$, let $a \in \mathcal{X}$ with base path γ and let \mathcal{E}_γ be the space of all paths β in T^*M of class C^2 with base path γ . Each such path induces a tangent vector in $T_a\mathcal{X}$, with components given by

$$X_\beta := (\pi^\sharp(\beta), \bar{\nabla}_a(\beta)) \in T_a\mathcal{X}.$$

With these, the foliation \mathcal{F} can be described as follows (see [7])

$$(24) \quad \mathcal{F}_a = \{X_\beta : \beta \in \mathcal{E}_\gamma, \beta(0) = 0, \beta(1) = 0\}.$$

Next, we give a very useful formula for Ω .

Lemma 5.3. *Let $a \in \mathcal{X}$ with base path γ . For $X = (\bar{X}, \theta_X)$, $Y = (\bar{Y}, \theta_Y) \in T_a\mathcal{X}$ choose $\beta_X, \beta_Y \in \mathcal{E}_\gamma$ such that $\theta_X = \bar{\nabla}_a(\beta_X)$ and $\theta_Y = \bar{\nabla}_a(\beta_Y)$. Then*

$$\Omega(X, Y) = \langle \beta_Y, \bar{X} \rangle|_0^1 - \langle \beta_X, \bar{Y} \rangle|_0^1 - \pi(\beta_X, \beta_Y)|_0^1.$$

In particular, for $Y = X_\beta$, with $\beta \in \mathcal{E}_\gamma$ we have that $\Omega(X, X_\beta) = \langle \beta, \bar{X} \rangle|_0^1$.

Proof. Using formulas (23) and (21), we compute:

$$\begin{aligned} \Omega(X, Y) &= \int_0^1 (\langle \theta_Y, \bar{X} \rangle - \langle \theta_X, \bar{Y} \rangle) dt = \int_0^1 (\langle \bar{\nabla}_a(\beta_Y), \bar{X} \rangle - \langle \bar{\nabla}_a(\beta_X), \bar{Y} \rangle) dt = \\ &= \int_0^1 \frac{d}{dt} (\langle \beta_Y, \bar{X} \rangle - \langle \beta_X, \bar{Y} \rangle) dt - \int_0^1 (\langle \beta_Y, \bar{\nabla}_a(\bar{X}) \rangle - \langle \beta_X, \bar{\nabla}_a(\bar{Y}) \rangle) dt = \\ &= \langle \beta_Y, \bar{X} \rangle|_0^1 - \langle \beta_X, \bar{Y} \rangle|_0^1 - \int_0^1 (\langle \beta_Y, \pi^\sharp(\theta_X) \rangle - \langle \beta_X, \pi^\sharp(\theta_Y) \rangle) dt, \end{aligned}$$

$$\begin{aligned}
\int_0^1 \langle \beta_Y, \pi^\sharp(\theta_X) \rangle dt &= \int_0^1 \langle \beta_Y, \pi^\sharp(\bar{\nabla}_a(\beta_X)) \rangle dt = \int_0^1 \langle \beta_Y, \bar{\nabla}_a(\pi^\sharp(\beta_X)) \rangle dt = \\
&= - \int_0^1 \langle \bar{\nabla}_a(\beta_Y), \pi^\sharp(\beta_X) \rangle dt + \int_0^1 \frac{d}{dt} (\langle \beta_Y, \pi^\sharp(\beta_X) \rangle) dt = \\
&= - \int_0^1 \langle \theta_Y, \pi^\sharp(\beta_X) \rangle dt + \langle \beta_Y, \pi^\sharp(\beta_X) \rangle|_0^1 = \int_0^1 \langle \beta_X, \pi^\sharp(\theta_Y) \rangle dt + \pi(\beta_X, \beta_Y)|_0^1.
\end{aligned}$$

□

Corollary 5.4. *Let $a \in \mathcal{X}$ with base path γ . Then*

$$\ker(\Omega_a) = \mathcal{F}_a = \{X_\beta : \beta \in \mathcal{E}_\gamma, \beta(0) = 0, \beta(1) = 0\}.$$

Proof. Consider $X = (\bar{X}, \theta_X) \in \ker(\Omega_a)$. It follows that for all $\xi \in \mathcal{E}_\gamma$ we have that $\Omega_a(X, X_\xi) = 0$, hence by the previous lemma $\bar{X}(0) = 0$ and $\bar{X}(1) = 0$. Let $\beta \in \mathcal{E}_\gamma$ be the unique solution to the equation $\theta_X = \bar{\nabla}_a(\beta)$ with $\beta(0) = 0$. Observe that by (21), both \bar{X} and $\pi^\sharp(\beta)$ satisfy the equation

$$\bar{\nabla}_a(Z) = \pi^\sharp(\theta_X), \quad Z(0) = 0.$$

Therefore they must be equal, and thus $X = X_\beta$. So, again by the lemma, for all $Y = (\bar{Y}, \theta_Y) \in T_a\mathcal{X}$, we have that $\langle \bar{Y}(1), \beta(1) \rangle = 0$. On the other hand, $Y(1)$ can be choose arbitrarily (see the Lemma 5.5 below), thus $\beta(1) = 0$ and this shows that $X \in \mathcal{F}_a$. The other inclusion follows directly from Lemma 5.3. □

Consider now the maps $\tilde{s}, \tilde{t} : \mathcal{X} \rightarrow M$ which assign to a path a the starting (respectively ending) point of its base path γ .

Lemma 5.5. *\tilde{s} and \tilde{t} are submersions and their fibers are orthogonal with respect to Ω . More precisely, denoting by \perp the orthogonal with respect to Ω , we have that*

$$(\ker d\tilde{s}_a)^\perp = \ker d\tilde{t}_a, \quad (\ker d\tilde{t}_a)^\perp = \ker d\tilde{s}_a.$$

Proof. To prove the first part, note that

$$\ker d\tilde{s}_a = \{(\bar{X}, \theta_X) \in T_a\mathcal{X} : \bar{X}(0) = 0\}, \quad \ker d\tilde{t}_a = \{(\bar{X}, \theta_X) \in T_a\mathcal{X} : \bar{X}(1) = 0\}.$$

Then, for $V_0 \in T_{\gamma(0)}M$, the path \bar{V} above γ which satisfies

$$\bar{\nabla}_a(\bar{V}) = 0, \quad \bar{V}(0) = V_0$$

induces $V = (\bar{V}, 0) \in T_a\mathcal{X}$ with $d\tilde{s}_a(V) = V_0$; so \tilde{s} , and similarly \tilde{t} , is a submersion.

For $X = (\bar{X}, \theta_X) \in \ker d\tilde{t}_a$, $Y = (\bar{Y}, \theta_Y) \in \ker d\tilde{s}_a$, let $\beta_X, \beta_Y \in \mathcal{E}_\gamma$ be so that

$$\bar{\nabla}_a(\beta_X) = \theta_X, \quad \beta_X(1) = 0, \quad \bar{\nabla}_a(\beta_Y) = \theta_Y, \quad \beta_Y(0) = 0.$$

Lemma 5.3 implies that $\Omega_a(X, Y) = 0$. Conversely, let $X = (\bar{X}, \theta_X) \in (\ker d\tilde{s}_a)^\perp$. For all $\xi \in \mathcal{E}_\gamma$, such that $\xi(0) = 0$ we have that $X_\xi \in \ker d\tilde{s}_a$, therefore, by assumption, $\Omega_a(X, X_\xi) = 0$. Thus, by Lemma 5.3 we have that

$$0 = \Omega_a(X, X_\xi) = \langle \xi(1), \bar{X}(1) \rangle.$$

But $\xi(1)$ is arbitrary, hence $\bar{X}(1) = 0$, i.e. $X \in \ker d\tilde{t}_a$. So $(\ker d\tilde{s}_a)^\perp = \ker d\tilde{t}_a$. □

We collect the main properties of Ω that are needed in the next subsection.

Proposition 5.6. *Let \mathcal{T} be a transversal to \mathcal{F} . Then the following hold:*

- (a) $\Omega|_{\mathcal{T}}$ is symplectic and is invariant under the holonomy action of \mathcal{F} on \mathcal{T} .
- (b) The sets $U_s = \tilde{s}(\mathcal{T})$ and $U_t = \tilde{t}(\mathcal{T})$ are open in M , and

$$\sigma = \tilde{s}|_{\mathcal{T}} : (\mathcal{T}, \Omega|_{\mathcal{T}}) \rightarrow (U_s, \pi|_{U_s}) \text{ is a Poisson map and}$$

$$\tau = \tilde{t}|_{\mathcal{T}} : (\mathcal{T}, \Omega|_{\mathcal{T}}) \rightarrow (U_t, \pi|_{U_t}) \text{ is anti-Poisson.}$$

- (c) $\ker(\sigma)^\perp = \ker(\tau)$ and $\ker(\tau)^\perp = \ker(\sigma)$.

Proof. Since \mathcal{F} is of finite codimension [7], there are no issues regarding the meaning of symplectic forms on our T and $\Omega|_{\mathcal{T}}$ is clearly symplectic. Actually, the entire (a) is a standard fact about kernels of closed two-forms, at least in the finite dimensions; it applies to our situation as well: from the construction of holonomy by patching together foliation charts, the second part is a local issue: given a product $B \times \mathcal{T}$ of a ball B in a Banach space and a finite dimensional manifold \mathcal{T} (for us a small ball in an Euclidean space) and a closed two-form Ω on $B \times \mathcal{T}$, if

$$\ker(\Omega_{x,y}) = T_x B \times \{0_y\} \subset T_x B \times T_y \mathcal{T}, \quad \forall (x, y) \in B \times \mathcal{T},$$

then $\Omega_x = \Omega|_{\{x\} \times \mathcal{T}} \in \Omega^2(\mathcal{T})$ does not depend on $x \in B$ (since Ω is closed).

For part (b) we will prove the statement for σ , for τ it follows similarly. Since

$$d\tilde{s}_a : T_a \mathcal{X} = T_a \mathcal{T} \oplus \mathcal{F}_a \rightarrow T_{\gamma(0)} M$$

is surjective and $\mathcal{F}_a \subset \ker d\tilde{s}_a$, it follows that $\tilde{s}|_{\mathcal{T}}$ is a submersion onto the open $\tilde{s}(\mathcal{T}) = U_s$. To show that σ is a symplectic realization, we will describe the Hamiltonian vector fields of $\sigma^*(f)$, for $f \in C^\infty(U_s)$. Consider the vector field on \mathcal{X} :

$$\tilde{H}_{f,a} := X_{(1-t)df_{\gamma(t)}} = (\bar{\nabla}_a((1-t)df_{\gamma(t)}), (1-t)\pi^\sharp(df_{\gamma(t)})) \in T_a \mathcal{X}.$$

Then we have that $d\tilde{s}(\tilde{H}_f) = \pi^\sharp(df)$, and by Lemma 5.3 it also satisfies

$$\Omega(\tilde{H}_f, Y)_a = \langle df_{\gamma(0)}, \bar{Y}(0) \rangle = d(\tilde{s}^* f)(Y), \quad (\forall) Y \in T_a \mathcal{X}.$$

Thus $\Omega(\tilde{H}_f, \cdot) = d(\tilde{s}^* f)$. Decomposing $\tilde{H}_f|_{\mathcal{T}} := H_f + V_f$, where H_f is tangent to \mathcal{T} and V_f is tangent to \mathcal{F} and using the fact that $\mathcal{F} = \ker \Omega$, it follows that

$$\Omega|_{\mathcal{T}}(H_f, \cdot) = d(\sigma^* f), \quad d\sigma(H_f) = \pi^\sharp(df).$$

This shows that σ is Poisson. Part (c) follows from Lemma 5.5 and Corollary 5.4. \square

5.4. Step 2.3: the needed symplectic realization. Back to the main theorem, to finish the proof, we still have to prove the existence of a symplectic realization as in Theorem 5.2. We will do that using the methods from the previous subsection; in particular, we keep the same notations. We consider:

- $\mathcal{Y} = \tilde{s}^{-1}(S) \subset \mathcal{X}$, the submanifold of \mathcal{X} sitting above S . Note that this is the same as the manifold $P(A)$ of A -paths of the algebroid $A = A_S$.
- The restriction of \mathcal{F} to \mathcal{Y} , $\mathcal{F}_{\mathcal{Y}} = \mathcal{F}|_{\mathcal{Y}}$. Again, this is the foliation $\mathcal{F}(A)$ associated to the algebroid A [7], and \mathcal{Y}/\mathcal{F} is the groupoid $\mathcal{G}(A)$ of A .

From the assumptions of the theorem, $\mathcal{G}(A)$ is compact. We denote it by B here.

As in the appendix in [11], we will use the following technical lemma:

Proposition 5.7. *Let \mathcal{F} be a foliation of finite codimension on a Banach manifold \mathcal{X} and let $\mathcal{Y} \subset \mathcal{X}$ be a submanifold which is saturated with respect to \mathcal{F} (i.e., each leaf of \mathcal{F} which hits \mathcal{Y} is contained in \mathcal{Y}). Assume that:*

- (H0) *The holonomy groups of the foliation \mathcal{F} at the points of \mathcal{Y} are trivial.*
- (H1) *$\mathcal{F}_{\mathcal{Y}} := \mathcal{F}|_{\mathcal{Y}}$ is induced by a submersion $p : \mathcal{Y} \rightarrow B$, with B -compact.*
- (H2) *The fibration $p : \mathcal{Y} \rightarrow B$ is locally trivial.*

Then one can find:

- (i) *a transversal $\mathcal{T} \subset \mathcal{X}$ to the foliation \mathcal{F} such that $\mathcal{T}_{\mathcal{Y}} := \mathcal{Y} \cap \mathcal{T}$ is a complete transversal to $\mathcal{F}_{\mathcal{Y}}$ (i.e., intersects each leaf of $\mathcal{F}_{\mathcal{Y}}$ at least once).*
- (ii) *a retraction $r : \mathcal{T} \rightarrow \mathcal{T}_{\mathcal{Y}}$.*
- (iii) *an action of the holonomy of $\mathcal{F}_{\mathcal{Y}}$ on $r : \mathcal{T} \rightarrow \mathcal{T}_{\mathcal{Y}}$ along \mathcal{F} .*

Moreover, the quotient of \mathcal{T} by the action of $\mathcal{F}_{\mathcal{Y}}$ is a smooth (Hausdorff) manifold.

In our case, once we make sure that the lemma can be applied, the resulting quotient Σ of \mathcal{T} will produce the desired symplectic realization. This follows from Proposition 5.6 and the fact that, by construction, $\sigma^{-1}(S) = \mathcal{G}(A) \subset \Sigma$.

(H1) is clear since P_x is smooth and compact. For (H2), we need the following:

Lemma 5.8. *Let M be a finite dimensional manifold, $x_0 \in M$, and denote by $\text{Path}(M, x_0)$ the Banach manifold of C^2 paths in M starting at x_0 . Then*

$$\epsilon : \text{Path}(M, x_0) \longrightarrow M, \quad \gamma \mapsto \gamma(1)$$

is a locally trivial fiber bundle.

Proof. Consider $x \in M$. For $U \subset V$ a small enough open neighborhoods of $x \in M$, we will construct a smooth family of diffeomorphisms

$$\phi_{y,t} : M \longrightarrow M, \text{ for } y \in U, t \in \mathbb{R},$$

such that $\phi_{y,t}$ is supported inside V , $\phi_{y,0} = \text{id}_M$ and $\phi_{y,1}(x) = y$. Then the required trivialization over U is given by

$$\tau_U : \epsilon^{-1}(x) \times U \longrightarrow \epsilon^{-1}(U), \quad \tau_U(\gamma, y)(t) = \phi_{y,t}(\gamma(t)),$$

with inverse

$$\tau_U^{-1}(\gamma)(t) = (\phi_{\gamma(1),t}^{-1}(\gamma(t)), \gamma(1)).$$

The construction of such diffeomorphisms is clearly a local issue, thus we may assume that $M = \mathbb{R}^m$, with $x = 0$ and $U = B_1(0)$, $V = B_2(0)$, the balls of radii 1 and 2 respectively. Consider $f \in C^\infty(\mathbb{R}^m)$, supported inside $B_2(0)$, with $f|_{B_1(0)} = 1$. Let $\phi_{y,t}$ be the flow at time t of the compactly supported vector field $X_y := f \cdot \vec{y}$, where \vec{y} represents the constant vector field on \mathbb{R}^m corresponding to $y \in B_1(0)$. Then it is easy to see that $\phi_{y,t}$ satisfies all requirements. \square

Next, for a groupoid \mathcal{G} over a manifold S , we denote by $\text{Path}^s(\mathcal{G}, 1)$ the Banach manifold of C^2 -paths γ in \mathcal{G} starting at some unit 1_x and satisfying $s \circ \gamma = x$. For $\mathcal{G} = \mathcal{G}(A)$, Proposition 1.1 of [7] identifies our bundle $p : \mathcal{Y} \longrightarrow B$ with the bundle

$$(25) \quad \tilde{\epsilon} : \text{Path}^s(\mathcal{G}, 1) \longrightarrow \mathcal{G}, \quad \gamma \mapsto \gamma(1).$$

Hence the following implies (H2).

Lemma 5.9. *For a source locally trivial Lie groupoid \mathcal{G} , the map $\tilde{\epsilon}$ (25) is a locally trivial fiber bundle.*

Proof. Consider $g_0 \in \mathcal{G}$, $x_0 = s(g_0)$. Consider a local trivialization of s over $U \ni x_0$, $\tau : s^{-1}(U) \cong U \times s^{-1}(x_0)$. Since the unit map is transversal to s , we may assume that $\tau(1_x) = (x, 1_{x_0})$ for all $x \in U$. Left composing with τ induces a diffeomorphism $\tau_* : \tilde{\epsilon}^{-1}(s^{-1}(U)) \longrightarrow U \times \text{Path}(s^{-1}(x_0), 1_{x_0})$, under which $\tilde{\epsilon}$ becomes

$$\text{Id} \times \epsilon : U \times \text{Path}(s^{-1}(x_0), 1_{x_0}) \longrightarrow U \times s^{-1}(x_0).$$

Since ϵ is trivial over $V \subset s^{-1}(x_0)$ around g_0 , $\tilde{\epsilon}$ is trivial over $\tau^{-1}(U \times V)$. \square

We still have to check (H0).

Lemma 5.10. *For any leaf \mathcal{L} of \mathcal{F} inside \mathcal{Y} , $\pi_1(\mathcal{L}) \cong \pi_2(P_x)$.*

Proof. The foliation $\mathcal{F}_{\mathcal{Y}}$ is given by the fibers of $p : \mathcal{Y} \longrightarrow B$, which, as remarked before, is isomorphic to the bundle $\tilde{\epsilon} : \text{Path}^s(\mathcal{G}, 1_y) \longrightarrow \mathcal{G}(A)$. So, a leaf \mathcal{L} will be identified with $\tilde{\epsilon}^{-1}(g)$, for some $g \in \mathcal{G}(A)$. Denote by $y := s(g)$ and $P_y := s^{-1}(y)$. Then \mathcal{L} is a fiber of $\epsilon : \text{Path}(P_y, y) \longrightarrow P_y$, which, by Lemma 5.8, is a locally trivial fiber bundle. Since $\text{Path}(P_y, 1_y)$ is contractible, by a standard argument, we find that $\pi_1(\mathcal{L}) \cong \pi_2(P_y)$. Since $\mathcal{G}(A)$ is transitive, P_y and P_x are diffeomorphic. \square

Of course, if $\pi_2(P_x)$ were assumed to be trivial, then condition (H0) follows automatically. Note that the hypothesis that $H^2(P_x) = 0$ is equivalent to $\pi_2(P_x)$ being finite; we show that this is enough to ensure triviality of the holonomy groups.

Lemma 5.11. *The holonomy group of the foliated manifold $(\mathcal{X}, \mathcal{F})$ is trivial at any point $a \in \mathcal{Y}$.*

Proof. Let $a \in \mathcal{Y}$ and let Γ be the holonomy group at a . Let \mathcal{T} be a transversal of $(\mathcal{X}, \mathcal{F})$ through a . Since Γ is finite, \mathcal{T} can be chosen small enough so that the holonomy transformations hol_u define an action of Γ on \mathcal{T} . Denote by $\mathcal{T}_\mathcal{Y} := \mathcal{T} \cap \mathcal{Y}$. Since the holonomy of $(\mathcal{Y}, \mathcal{F}_\mathcal{Y})$ is trivial, by making \mathcal{T} smaller, we may assume:

C1: the action of Γ on $\mathcal{T}_\mathcal{Y}$ is trivial.

Note also that the submersion $\sigma : \mathcal{T} \rightarrow M$ satisfies:

C2: σ is Γ -invariant.

C3: $\sigma^{-1}(\sigma(a)) \subset \mathcal{T}_\mathcal{Y}$.

C1 is clear. For C2: a' and $\text{hol}_u(a')$ are always in the same leaf, i.e. cotangent-homotopic, for $a' \in \mathcal{T}$, hence the starting point is the same. We have to show that the action of Γ is trivial in a neighborhood of a in \mathcal{T} . Since Γ is finite, it suffices to show that the induced infinitesimal action of Γ on $T_a\mathcal{T}$ is trivial. At the infinitesimal level, we have a short exact sequence

$$\ker(d\sigma)_a \xrightarrow{i} T_a\mathcal{T} \xrightarrow{(d\sigma)_a} T_xM.$$

This is a sequence of Γ -modules, where Γ acts trivially on the first and the last term. For the last map, this follows from C2. For the first map, C3 implies that $\ker(d\sigma)_a \subset T_a\mathcal{T}_\mathcal{Y}$ on which Γ acts trivially by C1. Since Γ is finite, the action on the middle term must be trivial as well (use e.g. an equivariant splitting). \square

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